Hybrid Mathematical Model of Inventory System with Piecewise Holding Cost and Its Optimal Strategy

Sutrisno
Department of Mathematics
Diponegoro University
Semarang, Indonesia
tresno.math@undip.ac.id,
tresnowijoyo@gmail.com

Widowati
Department of Mathematics
Diponegoro University
Semarang, Indonesia
widowati_math@undip.ac.id,
wiwied_wido@yahoo.com

Dita Anies Munawwaroh
Department of Mathematics
Diponegoro University
Semarang, Indonesia
dita.anies.m@gmail.com

***Abstract*—In this paper, we formulate a hybrid mathematical model of an inventory system with piecewise holding cost. We write the hybrid model of this system in the piecewise-affine (PWA) form. We use model predictive control method to generate the optimal strategy i.e. the amount of the arriving product shipment so that the inventory level tracks some desired inventory level given by decision maker as close as possible and this inventory system meets the demand with minimum total holding cost. Firstly, we convert the model of the PWA form into mixed logical dynamic (MLD) form and then we apply the model predictive control for hybrid system to this model in the MLD form. We simulate the model with various desired inventory level. From the computational simulation results, we observed that the inventory level follows the given desired inventory level very well.**

***Keywords*—inventory system; piecewise holding cost; hybrid mathematical model**

I. INTRODUCTION

Management on supply chain to reduce the operational cost and improve the profit has been developed by many researchers. Reference [1] was said that supply chain management focuses in relationships in order to maximize outcome for all parties in the chain. Supply chain network consisting of all parties that are manufacturer, suppliers, carriers or transporters, warehouses, retailers and customers [2]. The optimal strategy in supply chain management is made to minimize the operational cost with some service level constraint and determine the optimal quantity, optimal locations and optimal time of a product to be produced and distributed [3]. One of the parties in the supply chain management is inventory system that has to be controlled so that it gives the minimal holding cost. Inventory problem can appear in the several forms like raw material inventory, work-in-process inventory and finished product inventory and each of them needs to be controlled in order to minimize the holding cost [3].

To control an inventory system such that it will meet the demand and located on some desired inventory level given by decision maker, the dynamic of the inventory level or stock level can be modeled into a mathematical model. Mathematical model of inventory system and its strategy have been developed by several researchers to control the inventory level so that the minimal total holding cost will be achieved. References [4, 5, 6, 7, 8] were give the basic mathematical model of the inventory problem and its optimal policy to control the stock level. References [9, 10] was developing the basic inventory model by integrating it with supplier selection problem in their strategy. Some optimization methods were applied to control the inventory system like particle swarm optimization appeared in [11] and linear quadratic regulator method appeared in [4].

In system and control theory, hybrid dynamical system consists of discrete variable, continuous variable, real value variables, Boolean relation, if-then-else condition, on/off condition or their interaction which can be called as mode [12, 13]. A special case of hybrid dynamical system whose mode depends on the current location (or event) of the state vector can be called piecewise-affine (PWA) system that can be transformed into equivalent mixed logical dynamical system (MLD). The conversion from PWA into MLD and vice versa can be done by inputting the PWA model in hybrid systems description language (HYSDEL) and generating equivalent MLD by using function *mld* in hybrid toolbox for MATLAB given by [14]. To control a hybrid system for state regulator or track reference purposes, we can use model predictive control method for hybrid system that was designed in [12, 15] by following model predictive controller for linear system given by [16]. The formulation steps are predicting the state vector, input vector, auxiliary variables and output vector from MLD model, substituting them into a quadratic objective function and minimizing it by using some optimization method [12]. By solving the corresponding optimization problem, it will generate the optimal input or strategy for this system.

In the previous researches, the holding cost per unit is fixed wherever the stock level is. If the holding cost is different for several stock level intervals namely piecewise holding cost, we have to modify the older mathematical model of the inventory system in order to be controlled. In this paper, we formulate the hybrid mathematical model of an inventory system with piecewise holding cost. This model will be in the PWA form and we convert it into MLD model. Then we control this MLD model by generating the optimal strategy using model predictive control method so that the stock level follows a desired stock level as the reference decided by decision maker. Computational simulation will be given to observe the model and its optimal strategy work and compare the dynamic of the stock level with the desired stock level.

II. Hybrid Mathematical Model of Inventory Systems with Piecewise Holding Cost

Hybrid mathematical model of inventory system with piecewise holding cost is constituted by the event state i.e. the stock level of this inventory system. This event state is triggered by the difference of the holding cost for each of two or more stock level intervals. Consider an inventory system where the dynamic of the stock level depends on the arriving shipments and its demand. Let k denotes the time instant or time period of review on this inventory system, $y(k) \geq 0$ denotes the stock/inventory level at time period k and $d(k)$ denotes the demand for time period k.. Assumed that the inventory level is initially empty i.e. for $k < 0$ the inventory level $y(k) = 0$. Let $u(k)$ be the amount of arriving shipment of the product that arrived at review period k and n_p be the lead time delay, then the dynamic of y for any $k \geq 0$, that is the dynamic of the stock level for some stock level interval may be expressed as

$$\begin{aligned} y(k) &= u(0) + u(1) + \ldots + u(k - n_p - 1) \\ &\quad - d(0) - d(1) - \ldots - d(k-1) \\ &= \sum_{j=0}^{k-n_p-1} u(j) - \sum_{j=0}^{k-1} d(j). \end{aligned} \tag{1}$$

Let $y(k) = x_1(k)$ and $x_i(k) = q_j u(k - n_p + i)$ for $i = 2,3,\ldots,n_p+1$ where $q_j, j = 1,2,\ldots,m$ denotes the holding cost for stock level interval $\hat{y}_0 = 0 \leq y(k) \leq \hat{y}_1$ and $\hat{y}_{j-1} < y(k) \leq \hat{y}_j$ for $j = 2,3,\ldots,m$, then the dynamic (1) for stock level interval $\hat{y}_{j-1} < y(k) \leq \hat{y}_j$ may be expressed as

$$\left.\begin{aligned} x_1(k+1) &= x_1(k) + x_2(k) - q_j d(k) \\ x_2(k+1) &= x_3(k) \\ x_3(k+1) &= x_4(k) \\ &\vdots \\ x_{(n-1)}(k+1) &= x_n(k) \\ x_n(k+1) &= q_j u(k). \end{aligned}\right\} \tag{2}$$

where $n = n_p + 1$. Let $x(k) = \left[x_1(k), x_2(k), \ldots, x_n(k)\right]'$ and then system (2) may be expressed as the following linear time invariant discrete time state space

$$x(k+1) = Ax(k) + B_j u(k) + Dd(k) \tag{3}$$

where $A = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$, $B_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ q_j \end{bmatrix}$, $D = \begin{bmatrix} -q_j \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

Then the hybrid mathematical model of this inventory system with piecewise holding cost may be written as the following PWA model

$$x(k+1) = \begin{cases} Ax(k) + B_1 u(k) + Dd(k) & \text{if } 0 = \hat{y}_0 \leq y(k) \leq \hat{y}_1 \\ Ax(k) + B_2 u(k) + Dd(k) & \text{if } \hat{y}_1 < y(k) \leq \hat{y}_2 \\ \vdots & \\ Ax(k) + B_m u(k) + Dd(k) & \text{if } \hat{y}_{m-1} < y(k) \leq \hat{y}_m = \hat{y}_{\max} \end{cases} \tag{4}$$

$$y(k) = \begin{cases} \frac{1}{q_1} x(k), \text{ if } \hat{y}_0 < y(k) \leq \hat{y}_1 \\ \vdots \\ \frac{1}{q_j} x(k), \text{ if } \hat{y}_{j-1} < y(k) \leq \hat{y}_j \\ \vdots \\ \frac{1}{q_m} x(k), \text{ if } \hat{y}_{m-1} < y(k) \leq \hat{y}_m \end{cases} \tag{5}$$

Hybrid mathematical model (4)-(5) will be controlled such that the output state of this system i.e. the stock level $y(k)$ follows the desired stock level y_d as close as possible.

III. Model Predictive Control Method for Hybrid System

We will determine the optimal strategy for this inventory system by controlling PWA model (4)-(5) using model predictive control method. Firstly, we convert PWA model (4) -(5) into MLD model by inputting (4)-(5) in HYSDEL. Then we generate the MLD model by using *mld* function in MATLAB hybrid control toolbox given by [17]. It will be in the form

$$x(k+1) = Ax(k) + B_1 u(k) + B_2 \delta(k) + B_3 z(k) \tag{6}$$

$$y(k) = Cx(k) + D_1 u(k) + D_2 \delta(k) + D_3 z(k) \tag{7}$$

$$E_2 \delta(k) + E_3 z(k) \leq E_1 u(k) + E_4 x(k) + E_5 \tag{8}$$

where z is real valued auxiliary variable and δ is binary auxiliary variables, A, B_1, B_2, B_3, C, D_1, D_2, D_3, E_1, E_2, E_3, E_4 and E_5 are real constant matrices with appropriate dimension. This MLD model is another mathematical model representation for this inventory system. Model predictive control method for MLD system can be applied for regulator or reference/trajectory tracking purposes. In this paper, we will use it for trajectory tracking purpose where system (6)-(8) is assumed to be controllable and observable. Let H_p be the length of the horizon prediction, the objective function is defined as the gain of the actual stock level from the desired stock level with weighting matrix Q_1 and the input cost with weighting matrix Q_2. Then the optimal control problem for this inventory problem is

$$\min_{[u,\delta,z]_0^{H_p-1}} J\left([u,\delta,z]_0^{H_p-1},x_0\right)=\sum_{k=0}^{H_p-1}\left[\|y(k)-y_d\|_{Q_1}^2+\|u(k)\|_{Q_2}^2\right] \quad (9)$$

subject to :

$$\begin{cases} x(k+1)=Ax(k)+B_1u(k)+B_2\delta(k)+B_3z(k) \\ y(k)=Cx(k)+D_1u(k)+D_2\delta(k)+D_3z(k) \\ -E_4x(k)-E_1u(k)+E_2\delta(k)+E_3z(k)\le E_5 \\ u_{\min}\le u(k)\le u_{\max} \\ x_{\min}\le x(k)\le x_{\max} \\ y_{\min}\le y(k)\le y_{\max} \end{cases}$$

where $k=0,1,...,H_p-1$, matrices Q_1, Q_2 are the assumed to be symmetric and positive definite, notations $v_{\min}$ and $v_{\max}$ mean the lower bound and upper bound for v respectively and the notation $\|v\|_Q^2$ means v^TQv. Model predictive control method works by forming the prediction vectors for state, input and output for (6)-(8), substituting them into (9) and solving the corresponding optimization problem by using some optimization method. We solve (9) by using mixed integer quadratic programming (miqp). It generates the solution, denoted by superscript *, for u^*, δ^*, and z^*. The control action i.e. the optimal strategy for this inventory system is u^* at current time step k.

IV. Computational Simulation

Suppose that we have an inventory system with piecewise holding cost where the holding cost per unit for the first 100 units is \$20 and \$10 for the second 100 units. Let the lead time delay $n_p=2$ then n=3 and the dynamic of vector x is

$$\begin{cases} x_1(k+1)=x_1(k)+x_2(k)-q_jd(k) \\ x_2(k+1)=x_3(k) \\ x_3(k+1)=q_ju(k) \end{cases}$$

or

$$\begin{cases} x(k+1)=Ax(k)+B_ju(k)+Dd(k) \\ \quad y(k)=Cx(k) \end{cases}$$

where $A=\begin{bmatrix}1&1&0\\0&0&1\\0&0&0\end{bmatrix}, B_j=\begin{bmatrix}0\\0\\q_j\end{bmatrix}, D=\begin{bmatrix}-q_j\\0\\0\end{bmatrix}, C=\begin{bmatrix}1&0&0\end{bmatrix}$.

Since $q_1=\$20$ for $0=\hat{y}_0\le y(k)\le\hat{y}_1=100$ and $q_2=\$10$ for $101=\hat{y}_1\le y(k)\le\hat{y}_2=200$, then the matrices B_j for (3) are

$$B_1=\begin{bmatrix}0\\0\\\vdots\\20\end{bmatrix} \text{ and } B_2=\begin{bmatrix}0\\0\\\vdots\\10\end{bmatrix}.$$

Hence the hybrid mathematical model of this inventory system can be written as the following PWA model

$$x(k+1)=\begin{cases} Ax(k)+B_1u(k)+Dd(k) & \text{if } \quad 0\le y(k)\le 100 \\ Ax(k)+B_2u(k)+Dd(k) & \text{if } \ 100<y(k)\le 200 \end{cases} \quad (10)$$

$$y(k)=\begin{cases} \frac{1}{20}x(k), & \text{if } \quad 0\le y(k)\le 100 \\ \frac{1}{10}x(k), & \text{if } \ 100<y(k)\le 200 \end{cases} \quad (11)$$

We convert PWA model (10)-(11) into equivalent MLD model by using function *mld* in *hybrid control toolbox* for MATLAB. The corresponding matrices for this MLD model are appeared in appendix. This MLD model will be controlled by determining its optimal input such that the output of this system i.e. the stock level $y(k)$ follows the desired stock level y_d as closed as possible. The value of the desired stock level can be found in the simulation result. The demand for this simulation is assumed to be constant that is $d(k)=5$ for all $k>0$. The parameter values for this simulation are given by TABLE I.

TABLE I. Parameter Values

Parameter	Q_1, Q_2	H_p	$u_{\min}$	$u_{\max}$	$y_{\min}$	$y_{\max}$
Value	I^a	10	0	100	0	200

a. Identity matrix with appropriate dimension

We simulate this system with 80 review periods. The simulation results are shown by Fig. 1-3. The optimal input u^* is shown by Fig. 1. The actual stock level $y(k)$ and the desired stock level $y_d(k)$ are shown by Fig. 2. The value for artificial binary variable δ^* is shown by Fig. 3.

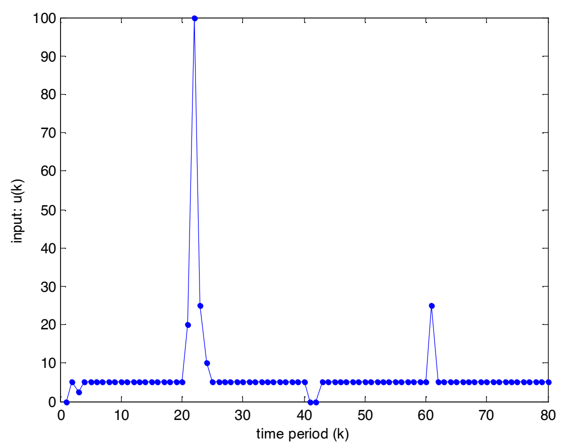

Fig. 1. Optimal input $u(k)$

Fig. 1 shows the optimal input u i.e. the amount of the arriving shipment of the product for each time or review period k that was generated by optimization (9). From this figure, it can be seen that from time period 0 to 20, the shipment is sufficiently small that is about 5 items, but at time period 22 the product shipment is 100 items due to the increasing of the desired inventory level is sufficiently large i.e. from 95 items to 110 items and it also has to meet the demand at current time period. After time period 25 and so on the product shipment is relatively constant due to the demand is constant. This optimal input shown by Fig. 1 is applied to the system to generate the output i.e. the actual stock level $y(k)$ shown by Fig. 2.

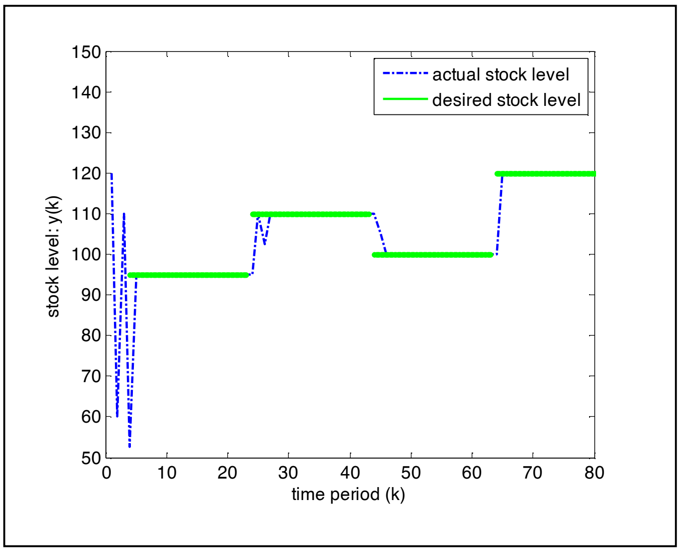

Fig. 2. Output: actual stock level $y(k)$ and desired stock level y_d

Fig. 2 shows the output of this inventory system i.e. the stock level (actual) and the desired stock level decided by decision maker. From this figure, it can be seen the at time period 0 to 5, the stock level is fluctuated due to the initial stock level is 120 items, the demand is 5 items and the desired inventory level is 95 items, so the controller must calculate input value that is the product shipment for current time period so that the stock meets the demand and follows the desired stock level simultaneously with minimum holding cost. It also can be seen that at each time period 5 to 80, the inventory level is sufficiently close to the desired stock level.

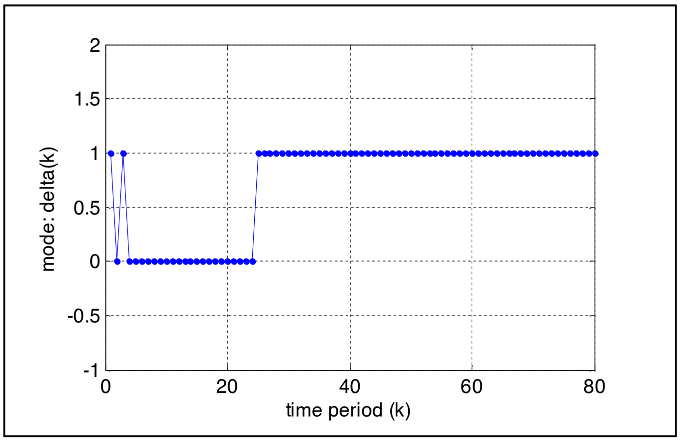

Fig. 3. Dynamic of the mode

Fig. 3 shows the additional information that is the mode value for this inventory system. Mode 0 means the stock level y is located on interval $[0,100]$ and mode 1 means the stock level y is located on interval $(100,200]$. This mode value for each time period is follows the PWA model (11) where the first part of equation (11) is related to mode-0 and the second part is related to mode-1.

V. Conclusion And Future Research

In this paper, control problem of inventory system with piecewise holding cost was considered. The hybrid mathematical model of this system was formulated as a piecewise-affine model. The model predictive control method was applied to generate the optimal strategy i.e. the amount of the arriving shipment so that the output i.e. the stock level follows the desired stock level given by decision maker. From simulation result, it can be conclude that the stock level of this inventory system was followed the desired stock level well.

In the future research, we will simulate the model if the demand is non-stationary and also if the demand is uncertain. Furthermore, we will develop this hybrid mathematical model by integrating it with supplier selection problem, hence we will find the optimal strategy for the inventory control problem and supplier selection problem simultaneously.

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APPENDIX

Matrices for MLD (6)-(8)

Matrix	A	B_1	B_2	B_3	C	D_1	D_2	D_3
Values	[0 0 0 0	[0	[0	[1 0 0 0	[0 0 0 1]	0	0	[0 0 0 0]
	0 0 0 0	0	0	0 1 0 0				
	0 0 0 0	0	0	0 0 1 0				
	0 0 0 0]	0]	0]	0 0 0 1]				

Matrix	E_1	E_2	E_3	E_4	E_5
Values	[0	1.0e+03 *	[0 0 0 0	[0 0 0 -1.0000	[1.0e+03 *
	0		0 0 0 0	0 0 0 1.0000	
	0	[-0.1000	-1 0 0 0	-1.0000 -1.0000 0 0	0.1000
	0	0.1000	1 0 0 0	1.0000 1.0000 0 0	0
	0	7.9500	-1 0 0 0	-1.0000 -1.0000 0 0	8.0500
	0	8.0500	1 0 0 0	1.0000 1.0000 0 0	7.9500
	0	-8.0500	0 -1 0 0	0 0 -1.0000 0	0.0500
	0	-7.9500	0 1 0 0	0 0 1.0000 0	-0.0500
	0	4.0000	0 -1 0 0	0 0 -1.0000 0	4.0000
	0	4.0000	0 1 0 0	0 0 1.0000 0	4.0000
	-20	-4.0000	0 0 -1 0	0 0 0 0	0
	20	-4.0000	0 0 1 0	0 0 0 0	0
	-10	2.0000	0 0 -1 0	0 0 0 0	2.0000
	10	1.0000	0 0 1 0	0 0 0 0	1.0000
	0	-1.0000	0 0 0 -1	-0.0500 0 0 0	0
	0	-2.0000	0 0 0 1	0.0500 0 0 0	0
	0	0.2000	0 0 0 -1	-0.1000 0 0 0	0.2000
	0]	0.4000	0 0 0 1]	0.1000 0 0 0]	0.4000
		-0.4000			0
		-0.2000]			0]