Stock Control of Single Product Inventory System with Imperfect Delivery by Using Robust Linear Quadratic Regulator

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Abstract—**In advanced inventory management problem, the received product will be not always entirely accepted due to the imperfect delivery process. The accepted delivered product volume can be represented by acceptance rate that is unknown/uncertain for the future delivery process. Then, a mathematical model and control method that can handle this problem is needed in order to control the product stock level in the warehouse. In this paper, we formulate a dynamical linear system with random parameter in the matrix coefficient to solve the inventory control problem with imperfect delivery proves and apply the robust linear quadratic regulator (RLQR) to find the optimal decision i.e. the optimal volume of the product that will be purchased at any review time period so that the inventory level will follow some set point decided by the decision maker with minimal effort (cost). To illustrate the model and its optimal decision, a computational experiment was performed with randomly generated inventory data. From the result, the optimal volume of the product for each review time period was determined where the product stock level was followed the desired set point.**

Keywords— Inventory control, imperfect delivery, robust LQR

I. INTRODUCTION

Logistics and supply chain management involves determining the optimal strategy for managing the procurement, movement and storage of materials, parts and finished products [1]. Inventory system is one of many components on logistics management. Inventory level of a product has to be controlled for demand satisfying even if demand is following by stochastic parameter itself. There are 2 environment classes of demand value which are deterministic environment and stochastic environment. Many researchers were developed the mathematical model to solve the inventory control problem in deterministic environment. The basic dynamical system for inventory control for single product and single supplier case was formulated in [2] where the optimal decision was determined by using classic linear quadratic control method. The more advance model was formulated in a linear hybrid system in order to handle discount price [3].

Some cases have a more advance problem when the dynamical system has uncertain parameters which is called robust control problem. In the last decade, several methods have been offered to solve robust control problem, some of them are using iterative and Riccati equation that were discussed in [4]–[6]. Another approach is using robust predictive control discussed in [7]. To control a dynamical system involving uncertainties, in control theory, there are many advanced control methods that were developed from classical control methods such as robust linear quadratic regulator that was developed from classical linear quadratic regulator (RLQR) [8]. By following the first developed RLQR, it was extended to be a new developed control method like RLQR for switched system [9]. From many published research articles, RLQR was given a good performance and it was applied in many subjects like mechanical systems [10]–[13] and many other subjects.

In this paper, a dynamical system in a linear system with random coefficient matrices will be formulated to solve the inventory control problem with imperfect delivery process where the acceptance rate of the delivered product is approached as a random variable. We will use RLQR control method to determine the optimal control value that is the ordered product volume for each review time period so that the inventory level is satisfying the demand and it will follow a desired (reference) level as close as possible with minimal effort.

II. MATHEMATICAL MODEL

Given a single product inventory system with periodical review which state $y(k)\in\mathbb{R}$ denotes the on-hand stock (inventory/product stock level) at review time period $k=0,1,2,\ldots$. The dynamics of y at any review time period k depends on the amount of the arriving product shipment at *k*, the stored product at the previous review time period $(k-1)$ and on the demand value at *k* denoted by $d(k)\in\mathbb{R}$. By assuming that for $k<0$ there is no stored product then it means that $y(k)=0$ for $k<0$ and let the inventory level at review time period 0 is $y(0)=y_0$. Let $u(k)\in\mathbb{R}$ denotes the product amount that be ordered from the supplier at review time period k and $n_p\in\{0,1,2,\ldots\}$ denotes the lead time delay of the ordered product, and $c\in[0,1]$ be the acceptance rate of the

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delivered/received product. Then, the stock level for any $k > 0$ is formulated as

$$\begin{aligned} y(k) &= c.u(-n_p) + c.u(1-n_p) + c.u(2-n_p) + \ldots + c.u(k-1-n_p) \\ &\quad - d(0) - d(1) - d(2) - \ldots - d(k-1). \end{aligned} \tag{1}$$

Assumed that the demand is constant for each review time period denoted by d_0 units and let $x_0(k) = -d(k), x_0(0) = -d(0) = -d_0$. Let $x_1(k) = y(k)$ represented the product stock level at review time period k, and $x_i(k) = u(k - n_p + i)$ for any $i = 2, \ldots, n$ denotes a delayed input signal u. Then, (1) may be expressed as

$$\begin{cases} x_0(k+1) = x_0(k) \\ x_1(k+1) = x_1(k) + x_2(k) - d(k) = x_1(k) + x_2(k) + x_0(k) \\ x_2(k+1) = x_3(k) \\ \vdots \\ x_{n-1}(k+1) = x_n(k) \\ x_n(k+1) = c \cdot u(k) \\ y(k) = x_1(k). \end{cases} \tag{2}$$

Let $x(k) = [x_0(k), x_1(k), x_2(k), \ \ldots \ , x_n(k)]^T$ called the state vector of the system. Then, (2) can be rewritten as the following state space equation

$$x(k+1) = \begin{pmatrix} x_0(k+1) \\ x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{pmatrix} = \begin{pmatrix} x_0(k) \\ x_0(k) + x_1(k) + x_2(k) \\ x_3(k) \\ \vdots \\ x_n(k) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_0(\mathrm{k}) \\ x_1(k) \\ x_2(k) \\ x_3(k) \\ \vdots \\ x_n(k) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ c \end{pmatrix} u(k) \tag{3}$$

$$y(k) = (0,1,0,\ldots,0)\, x(k).$$

We describe previous system into a discrete-time linear system with parametric uncertainties as

$$x(k+1) = \{F + \delta F\} x(k) + \{G + \delta G\} u(k), \ k = 0,1,2,\ldots \tag{4}$$

where

$$F = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \delta G = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ c \end{pmatrix},$$

$$G = 0_n, \delta F = \mathbf{0}, [\delta F, \delta G] = H\Delta[E_F, E_G],$$

matrices δF and δG are represented the uncertainty matrices in which $H \in \mathbb{R}^{n\times 1}, E_F \in \mathbb{R}^{1\times n}, E_G \in \mathbb{R}^{1\times m}$ are known matrices, and $-1 \le \Delta \le 1$ is arbitrary. It is assumed that matrix H is a nonzero.

III. Robust Linear Quadratic Regulator Review

Given system (4) with assumption

$$rank\left(\begin{bmatrix} E_{F_u} & E_{G_i} \end{bmatrix}\right) = rank\left(E_{G_i}\right)$$

is hold. We will apply the existing optimal control method called robust linear quadratic regulator (RLQR) which minimizes [8]

$$\min_{x(k+1),u(k)} \max_{\delta F, \delta G} \left\{ J^{\mu}\left(x(k+1), u(k), \delta F, \delta G\right) \right\} \tag{5}$$

where the one-step quadratic cost function is given by

$$\begin{aligned} J^{\mu}\left(x(k+1), u(k), \delta F, \delta G\right) &= \begin{bmatrix} x(k+1) \\ u(k) \end{bmatrix}^T \begin{bmatrix} P(k+1) & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(k+1) \\ u(k) \end{bmatrix} \\ &+ \left\{ \left(\begin{bmatrix} 0 & 0 \\ I & -G \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\delta G \end{bmatrix} \right) \begin{bmatrix} x(k+1) \\ u(k) \end{bmatrix} - \left(\begin{bmatrix} -I \\ F \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ \delta F \end{bmatrix} x(k) \right) \right\}^T \\ &\begin{bmatrix} Q & 0 \\ 0 & \mu I \end{bmatrix} \left\{ \left(\begin{bmatrix} 0 & 0 \\ I & -G \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\delta G \end{bmatrix} \right) \begin{bmatrix} x(k+1) \\ u(k) \end{bmatrix} \right. \\ &\left. - \left(\begin{bmatrix} -I \\ F \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ \delta F \end{bmatrix} x(k) \right) \right\} \end{aligned} \tag{6}$$

where $Q \succ 0, R \succ 0, P(k+1) \succ 0$ are weighting matrices and $\mu > 0$ is a penalty parameter. The optimization (5), and (6) can be solved for $\mu \to +\infty$ by using RLQR algorithm given in [8] by setting up $x(0)$ and $P(N+1) \succ 0$ first. The solution can be derived from (8) for all $k = N, \ldots, 0$ by finding $L \in \mathbb{R}^{n\times n}$, $K \in \mathbb{R}^{m\times n}$, and $P \in \mathbb{R}^{n\times n}$ with backward iteration, i.e.

$$\begin{pmatrix}L\\K\\P\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\0&0&-I\\0&0&F\\0&0&E_F\\I&0&0\\0&I&0\end{pmatrix}^T \begin{pmatrix} P(k+1)^{-1}&0&0&0&0&I&0\\ 0&R^{-1}&0&0&0&0&I\\ 0&0&Q^{-1}&0&0&0&0\\ 0&0&0&0&0&I&-G\\ 0&0&0&0&0&0&-E_G\\ I&0&0&I&0&0&0\\ 0&I&0&-G^T&-E_G^{-1}&0&0 \end{pmatrix}^{-1} \begin{pmatrix}0\\0\\-I\\F\\E_F\\0\\0\end{pmatrix} \quad (7)$$

with $F, G, E_F, E_G, Q \succ 0,$ and $R \succ 0$ are known matrices obtained from the dynamical system of the plant. Furthermore, the optimal input and the corresponding state value are given by $x(k+1)^*$ and $u(k)^*$, for $k = 0,\dots,N$ where

$$\begin{pmatrix}x(k+1)^*\\u(k)^*\end{pmatrix} = \begin{pmatrix}L\\K\end{pmatrix} x(k)^* \quad (8)$$

and finally, from these results we can calculate the output of the system.

IV. Computational Experiment

Consider the inventory system with imperfect delivery with acceptance product rate c represented by system (4)-**Error! Reference source not found.** with the following parameter matrices and initial conditions:

$$F = \begin{pmatrix}1&0&0&0\\1&1&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}, G = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \quad \delta G = \begin{pmatrix}0\\0\\\vdots\\c\end{pmatrix},$$

$E_F = 0_{1\times 4}, E_G = (1), \delta F = \mathbf{0}_4,$ where the lead time delay $n_p = 3,$ demand value 20 units for any review time period, and $x_0 = (-20 \quad 50 \quad 5 \quad 10)^T$. The used weighting matrices are $P_{N+1} = I_4$, $Q = I_4$, $R = I_1$ and finally suppose that the desired set point (reference point) of the inventory level decided by the decision maker is 20 units. From (7) the derived RLQR matrix is

$$\begin{pmatrix}L\\K\\P\end{pmatrix} = \begin{pmatrix}1&0&0&0\\1&1&1&0\\0&0&0&1\\-1&0&-0.5&-0.5\\-1&0&-0.5&-0.5\\5&1&2&1\\1&2&1&0\\2&1&2.5&0.5\\1&0&0.5&2.5\end{pmatrix}$$

and we can calculate the state, input signal (ordered product), and inventory level by

$$x(k+1)^* = \begin{pmatrix}1&0&0&0\\1&1&1&0\\0&0&0&1\\-1&0&-0.5&-0.5\end{pmatrix} x(k)$$

$$u(k)^* = (-1 \quad 0 \quad -0.5 \quad -0.5)\, x(k)$$

$$y(k) = (0 \quad 1 \quad 0 \quad 0)\, x(k).$$

TABLE I shows the inventory level and input signal (ordered product units) for $k = 0,\dots,10$. It can be observed that the initial inventory level is 50 units stored product in the warehouse and for each review time period, amount of product following the first column in TABLE 1 is ordered but not all of them can be accepted to be stored in the warehouse due to damaging in the delivery process. The stored product (inventory level) is used to satisfy the demand for each review time period.

TABLE I. The Evolution of the Inventory Level and Input (Ordered Product Units)

k	$u(k)$	$y(k)$
0	-	50
1	35.0000	20.0000
2	25.0000	17.5000
3	25.0000	18.7500
4	22.5000	21.8750
5	21.2500	19.6875
6	23.1250	19.2188
7	22.8125	20.5469
8	22.0313	20.1172
9	22.5781	19.6680
10	22.6953	20.1074

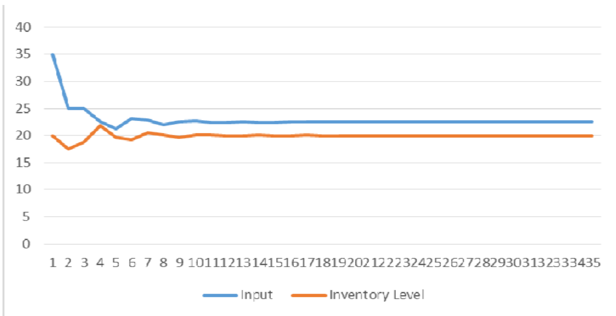

Fig. 1. Inventory level and Input

Fig. 1 shows the evolution of the product stock level and its optimal input value i.e. the ordered product. It can be seen that the optimal decision was derived by using the used RLQR control method and the inventory level followed the reference level 20 units. The stored product is used to satisfy the demand i.e. 20 for each review time period and the rest of them is stored in the warehouse to be used for the next review time period.

V. Conclusion

In this paper, an optimal control problem for single product inventory system with imperfect delivery was considered. The acceptance rate was desired as a random variable and the mathematical model was formulated in a linear system with uncertain matrix coefficient. The optimal decision was determined by applying RLQR control method. From the result of the computational experiment, it can be concluded that the optimal decision i.e. the optimal ordered product was derived for all given review time periods and the inventory level was followed the given set point (reference inventory level) well.

In our future works, we will develop the model for multi-supplier case and non-constant demand. Furthermore, we also will develop the model for multi-product inventory system. Comparison with other control methods such as robust predictive control and linear quadratic Gaussian will also be performed.

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