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Routh-hurwitz criterion and bifurcation method for stability analysis of tuberculosis transmission model

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Abstract Tuberculosis is an infectious disease; it caused by *Mycobacterium tuberculosis*. In this paper, we discuss how to use the Routh-Hurwitz stability criterion to analyze the stability of disease free of the tuberculosis transmission model. From this method, can be found the number of roots of the characteristic polynomial (eigenvalues) with positive real parts is equal to the number of changes in sign of the first column of the Routh array. If all of the eigenvalues are negative, then the model is stable. While the bifurcation method is used to analyze the stability of the endemic equilibrium point of the tuberculosis transmission, the endemic equilibrium point is locally asymptotically stable if reproduction number greater than one and additional parameters requirement that bifurcation met. Finally, numerical simulations are demonstrated to verify the used method.

1. Introduction

Tuberculosis (TB) is a disease caused by *Mycobacterium tuberculosis* [1]. The transmission of happened through which air is containing bacilli TB in sprinkling the spittle issued by victims pulmonary tuberculosis at the time of coughing, sneezing or singing. The treatment of TB becomes difficult because of multidrug-resistant tuberculosis cases. The treatment is longer and needs medicine that is more expensive. But for many developing countries are entirely free the maintenance is not. Some patients could not be capable of fully pay for medical treatment. Some patients who had yet to recover, choose to cast off treatment and choose to treat at home to save money [2].

Construction, analysis and the use of math model is regarded as one of the most important math. Model mathematics used in many fields and field of study different [3]. Some journal model has been studying math down into the real world especially world health, which is [4,5] studying about TB, [6] Studying about HIV/AIDS, [7] studying about syphilis, [8] studying about influenza and [9] studying about Ebola. The spreading of a TB disease in this divided into five subpopulations, namely susceptible class (S), *latent* class (E), infectious treated at home class (I_1), contagious treated at hospital class (I_2), dan recovery class (R) [10].

2. Routh-Hurwitz Criterion and Bifurcation Method

Routh-Hurwitz criterion [10] is a method to show the system stability by taking the coefficients of an equation characteristic without counting the roots. Suppose the equation characteristic:

$$p(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0$$

with a_j is a coefficient in a real number, $j = 1, 2, \ldots, n$.

Under condition the value of $a_0 > 0$, it can be said that $p(\lambda)$ stable if all the roots have the parts real negative. Hurwitz matrix from the equation is

$$H = \begin{pmatrix} a_1 & a_0 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & a_1 & \ldots & 0 \\ a_5 & a_4 & a_3 & a_2 & \ldots & 0 \\ a_7 & a_6 & a_5 & a_4 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & \ldots & \ldots & a_n \end{pmatrix}$$

All elements with subscript more than n or less than 0 can be replaced by zero. The principle of the essentials and must be found on stability is about the principle of subdeterminant polynomial.

$$D_1 = a_1, \quad D_2 = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}, \quad \ldots \quad D_n = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & a_1 & \ldots & 0 \\ a_5 & a_4 & a_3 & a_2 & \ldots & 0 \\ a_7 & a_6 & a_5 & a_4 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & \ldots & \ldots & a_n \end{vmatrix}.$$

Of some explanation above, can be used each criteria square, cubic, and quartic. Then it can be started from a quadratic equation, that is

$$p(\lambda) = a_0\lambda^2 + a_1\lambda + a_2$$

obtained $D_1 = a_1 > 0$ and $D_2 = \begin{vmatrix} a_1 & a_0 \\ 0 & a_2 \end{vmatrix} = a_1a_2 > 0.$

Quadratic equation $p(\lambda) = a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ Routh-Hurwitz criterion apply, i.e.

$$D_1 = a_1 > 0, \quad D_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} = a_1a_2 - a_0a_3 > 0, \quad \text{and} \quad D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{vmatrix} = a_3D_2 > 0.$$

Bifurcation [11] is a change in the solution of the equilibrium point caused by a change in the parameter value. The exchanges of stability move from one equilibrium point to another are called transcritical bifurcation. Transcritical bifurcation occurs when a system has an eigenvalue which in reality is zero. A nonlinear system when the system has eigenvalue in the real part is zero, so to analyze its stability, it can use the manifold center theory. Differential equation system with ϕ parameter defines as follows.

$$\frac{dx}{dt} = f(x,\phi),\ f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \text{ dan } f \in C^2(\mathbb{R}^n \times \mathbb{R}).$$

Without reducing the generality, it is assumed that zero is the equilibrium of the system above for all the values of ϕ the parameter, that is

$$f(0,\phi) \equiv 0,\ \forall \phi$$

Assumption

1. $A = D_x f(0,0) = \left(\frac{\partial f_i}{dx_j}(0,0)\right)$ Is the linearization matrix of the equilibrium system around **0** where ϕ is where is evaluated at **0**. Zero is a simple eigenvalue of A, and another eigenvalue of A has a negative real part.
2. The matrix A has the right eigenvector W, and the left eigenvector V corresponds to the eigenvalue of 0. Suppose f_k it is a component of f and

$$a = \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0) \text{ dan } b = \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \beta_2^*}(0,0).$$

The local dynamics of the system around **0** are entirely determined by a and b.

i. $a>0, b>0$. If $\phi<0$ with $|\phi| = 1$, 0 locally asymptotically stable and there is negative a then the equilibrium is locally asymptotically stable.

ii. $a<0, b<0$. If $\phi<0$ with $|\phi| = 1$, *0* bot stable; if $0<|\phi| = 1$, *0* not stable and there is positive a the equilibrium unstable.

iii. $a>0, b<0$. If $\phi<0$ with $|\phi| = 1$, *0* not stable and there is negative a change to positive a then the equilibrium is unstable.

iv. $a<0, b>0$. If ϕ *the* change from negative to positive, **0** its stability changes from stable to unstable. Based on this, negative A changes to positive a and the equilibrium changes from nstable to locally asymptotically stable.

The differences this paper and the existing model are instability of endemic equilibrium point, and we used the bifurcation method and locally stable, then the existing model used the Lyapunov method and globally stable.

3. Model Analysis

Model analysis of the spreading of a disease TB includes several aspects of that is figure out the points of equilibrium, the number of basic reproduction, locally stability of equilibrium points, and numerical simulation. Given the dynamic model of the spread of TB diseases [2] as follows:

$$\dot{S} = \delta - (\beta_1 I_1 + \beta_2 I_2)S - \mu S, \qquad \dot{E} = (\beta_1 I_1 + \beta_2 I_2)S - b_0 E,$$

$$\dot{I}_1 = \theta E + \omega_2 I_2 - b_1 I_1, \qquad \dot{I}_2 = \varepsilon E + \omega_1 I_1 - b_2 I_2.$$

with $b_0 = \theta + \varepsilon + \mu$, $b_1 = \omega_1 + k_1 + \mu + d_1$, dan $b_2 = \omega_2 + k_2 + \mu + d_2$.

3.1 Equilibrium Points

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Free equilibrium points of the spread dynamic of TB disease is $E_0:(S,E,I_1,I_2)=\left(\frac{\delta}{\mu},0,0,0\right)$. While the point of equilibrium endemic obtained by means elimination between equation on the model dynamic the spread of TB disease obtained point

$$S^*=\frac{\delta}{\mu\Re_0}, \qquad E^*=\frac{\delta}{b_0}\left(1-\frac{1}{\Re_0}\right),$$

$$I_1^*=\frac{\mu(\omega_2\varepsilon+b_2\theta)(\Re_0-1)}{\beta_1(b_2\theta+\omega_2\varepsilon)+\beta_2(\omega_1\theta+b_1\varepsilon)}, \qquad I_2^*=\frac{\mu(b_1\varepsilon+\omega_1\theta)(\Re_0-1)}{\beta_1(b_2\theta+\omega_2\varepsilon)+\beta_2(\omega_1\theta+b_1\varepsilon)}.$$

3.2 Reproduction Number Analysis

$$\mathsf{F}=\begin{bmatrix}\mathsf{F}_1\\ \mathsf{F}_2\\ \mathsf{F}_3\end{bmatrix}=\begin{bmatrix}(\beta_1I_1+\beta_2I_2)S\\ 0\\ 0\end{bmatrix} \text{ and } \mathsf{V}=\begin{bmatrix}\mathsf{V}_1\\ \mathsf{V}_2\\ \mathsf{V}_3\end{bmatrix}=\begin{bmatrix}b_0E\\ b_1I_1-\omega_2I_2-\theta E\\ b_2I_2-\omega_1I_1-\varepsilon E\end{bmatrix}.$$

Suppose F is Jacobian matrix from F and V is Jacobian matrix from V. So, at the free equilibrium disease $E_0(S,E,I_1,I_2)=\left(\frac{\delta}{\mu},0,0,0\right)$ obtained :

$$F(x)=\begin{bmatrix}0 & \frac{\beta_1\delta}{\mu} & \frac{\beta_2\delta}{\mu}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \text{ and } V(x)=\begin{bmatrix}b_0 & 0 & 0\\ -\theta & b_1 & -\omega_2\\ -\varepsilon & -\omega_1 & b_2\end{bmatrix}$$

Next Generation Matrix (NGM) obtained by the equation

$$NGM=FV^{-1}=\begin{bmatrix}\frac{\beta_1\delta(\theta b_2+\omega_2\varepsilon)+\beta_2\delta(\theta\omega_1+b_1\varepsilon)}{\mu b_0(b_1b_2-\omega_1\omega_2)} & \frac{\delta(\beta_1b_2+\beta_2\omega_1)}{\mu(b_1b_2-\omega_1\omega_2)} & \frac{\delta(\beta_1\omega_2+\beta_2b_1)}{\mu(b_1b_2-\omega_1\omega_2)}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$|\lambda I-NGM|=\left|\begin{bmatrix}\lambda-\frac{\beta_1\delta(\theta b_2+\omega_2\varepsilon)+\beta_2\delta(\theta\omega_1+b_1\varepsilon)}{\mu b_0(b_1b_2-\omega_1\omega_2)} & -\frac{\delta(\beta_1b_2+\beta_2\omega_1)}{\mu(b_1b_2-\omega_1\omega_2)} & -\frac{\delta(\beta_1\omega_2+\beta_2b_1)}{\mu(b_1b_2-\omega_1\omega_2)}\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix}\right|=0$$

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Obtained $\lambda_1 = 0$, $\lambda_2 = 0$ and $\lambda_3 = \dfrac{\delta\left(\beta_1\left(b_2\theta+\omega_2\varepsilon\right)+\beta_2\left(\omega_1\theta+b_1\varepsilon\right)\right)}{\mu b_0\left(b_1b_2-\omega_1\omega_2\right)}$.

$\Re_0$ Obtained from the absolute biggest of eigenvalues from NGM so $\Re_0 = \dfrac{\delta\left(\beta_1\left(b_2\theta+\omega_2\varepsilon\right)+\beta_2\left(\omega_1\theta+b_1\varepsilon\right)\right)}{\mu b_0\left(b_1b_2-\omega_1\omega_2\right)}$. If $\Re_0 < 1$ it means that every individual infected can transmit the disease to less than one new patients so that TB sickness cannot have developed in the population. If $\Re_0 > 1$ it means that individuals infected can transmit the disease to more than new one patients so that TB sickness can spreading in the population

3.3 Stability at The Free Equilibrium Point of Free Disease

Theorem 1. Equilibrium point of free disease $E_0 = \left(\dfrac{\delta}{\mu},0,0,0\right)$ will be local stability if $\Re_0 < 1$ and

$$\delta\left(b_0+b_1+b_2\right)\left(\beta_1\left(b_2\theta+\omega_2\varepsilon\right)+\beta_2\left(\omega_1\theta+b_1\varepsilon\right)\right)+\Re_0 > \Re_0 b_0\left(b_0+b_1+b_2\right)\left(\theta\delta\beta_1+\varepsilon\delta\beta_2\right)+1$$

To proof the stability from the free equilibrium of disease $E_0\left(S,E,I_1,I_2\right)$, linearization of the equation of the dynamic system by determining the Jacobian matrix.

The Jacobian matrix of $E_0 : (S,E,I_1,I_2) = \left(\dfrac{\delta}{\mu},0,0,0\right)$ follows.

$$J\left(E_0\right) = \begin{bmatrix} -\mu & 0 & -\dfrac{\beta_1\delta}{\mu} & -\dfrac{\beta_2\delta}{\mu} \\ 0 & -b_0 & \dfrac{\beta_1\delta}{\mu} & \dfrac{\beta_2\delta}{\mu} \\ 0 & \theta & -b_1 & \omega_2 \\ 0 & \varepsilon & \omega_1 & -b_2 \end{bmatrix}$$

Stability at the point of free equilibrium disease can be seen from the value of the eigen at matrix jacobian. Based on the Routh Hurwitz criterion, if all the parts of the value of real eigen Jacobian matrix are negative so E_0 locally asyptotically stable.

$$\left|\lambda I - J\left(E_0\right)\right| = 0$$

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -\mu & 0 & -\frac{\beta_1\delta}{\mu} & -\frac{\beta_2\delta}{\mu} \\ 0 & -b_0 & \frac{\beta_1\delta}{\mu} & \frac{\beta_2\delta}{\mu} \\ 0 & \theta & -b_1 & \omega_2 \\ 0 & \varepsilon & \omega_1 & -b_2 \end{bmatrix} \right| = 0$$

$$\left\| \begin{matrix} \lambda+\mu & 0 & \frac{\beta_1\delta}{\mu} & \frac{\beta_2\delta}{\mu} \\ 0 & \lambda+b_0 & -\frac{\beta_1\delta}{\mu} & -\frac{\beta_2\delta}{\mu} \\ 0 & -\theta & \lambda+b_1 & -\omega_2 \\ 0 & -\varepsilon & -\omega_1 & \lambda+b_2 \end{matrix} \right\| = 0$$

$$\frac{(\lambda+\mu)\begin{pmatrix} \lambda^3\mu + \lambda^2\mu(b_0+b_1+b_2) + \lambda\left(\mu(b_0b_1+b_0b_2+b_1b_2-\omega_1\omega_2) - \delta(\theta\beta_1+\varepsilon\beta_2)\right) \\ +\mu b_0(b_1b_2-\omega_1\omega_2) - \delta\beta_1(\theta b_2+\varepsilon\omega_2) - \delta\beta_2(\theta\omega_1+\varepsilon b_1) \end{pmatrix}}{\mu} = 0$$

The equation characteristic of the matrix jacobian shaped polynomial as follows.

$$(\lambda+\mu)\left(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3\right) = 0$$

with

$$a_1 = b_0 + b_1 + b_2$$

$$a_2 = \frac{\mu(b_0b_1+b_0b_2+b_1b_2-\omega_1\omega_2) - \delta(\theta\beta_1+\varepsilon\beta_2)}{\mu}$$

$$a_3 = \frac{\mu b_0(b_1b_2-\omega_1\omega_2) - \delta\beta_1(\theta b_2+\varepsilon\omega_2) - \delta\beta_2(\theta\omega_1+\varepsilon b_1)}{\mu}$$

Based on an equation, obtained the eigenvalue $\lambda_1 = -\mu$. Cause $\lambda_1 = -\mu < 0$, the reality of an eigenvalue is negative. According *to Routh Hurwitz* stability criterion, so two other values are if and only if $a_1, a_2, a_3 > 0$ and $a_1a_2 - a_3 > 0$. So, obtained

$$a_1 = b_0 + b_1 + b_2$$

Because of parameter θ, ε, μ, b_1, ω_1, k_1, d_1, ω_2, k_2, $d_2 > 0$ then it that.

$$a_1 = b_0 + b_1 + b_2 = (\theta+\varepsilon+\mu) + (\omega_1+k_1+d_1+\mu) + (\omega_2+k_2+d_2+\mu) > 0$$

$$a_2 = \frac{\mu(b_0b_1+b_0b_2+b_1b_2-\omega_1\omega_2) - \delta(\theta\beta_1+\varepsilon\beta_2)}{\mu} = b_0b_1 + b_0b_2 + \frac{\delta\left(\beta_1(b_2\theta+\omega_2\varepsilon) + \beta_2(\omega_1\theta+b_1\varepsilon)\right) - \Re_0 b_0(\theta\delta\beta_1+\varepsilon\delta\beta_2)}{\Re_0\mu b_0}$$

$a_2 > 0$ accomplished if

$$\delta\left(\beta_1(b_2\theta+\omega_2\varepsilon) + \beta_2(\omega_1\theta+b_1\varepsilon)\right) > \Re_0 b_0(\theta\delta\beta_1+\varepsilon\delta\beta_2)$$

$$a_3 = \frac{\mu b_0\left(b_1 b_2 - \omega_1\omega_2\right) - \delta\beta_1\left(\theta b_2 + \varepsilon\omega_2\right) - \delta\beta_2\left(\theta\omega_1 + \varepsilon b_1\right)}{\mu} = \frac{\delta}{\mu\Re_0}\left(\beta_1\left(b_2\theta + \omega_2\varepsilon\right) + \beta_2\left(\omega_1\theta + b_1\varepsilon\right)\right)\left(1-\Re_0\right)$$

$a_3 > 0$ accomplished if $\Re_0 < 1$

$$\begin{aligned} a_1 a_2 - a_3 &= \left(b_0 + b_1 + b_2\right)\left(\delta\left(\beta_1\left(b_2\theta + \omega_2\varepsilon\right) + \beta_2\left(\omega_1\theta + b_1\varepsilon\right)\right) - \Re_0 b_0\left(\theta\delta\beta_1 + \varepsilon\delta\beta_2\right)\right) - \left(1-\Re_0\right) \\ &= \delta\left(b_0 + b_1 + b_2\right)\left(\beta_1\left(b_2\theta + \omega_2\varepsilon\right) + \beta_2\left(\omega_1\theta + b_1\varepsilon\right)\right) + \Re_0 - \Re_0 b_0\left(b_0 + b_1 + b_2\right)\left(\theta\delta\beta_1 + \varepsilon\delta\beta_2\right) - 1 \end{aligned}$$

$a_1 a_2 - a_3 > 0$ accomplished if

$$\delta\left(b_0 + b_1 + b_2\right)\left(\beta_1\left(b_2\theta + \omega_2\varepsilon\right) + \beta_2\left(\omega_1\theta + b_1\varepsilon\right)\right) + \Re_0 > \Re_0 b_0\left(b_0 + b_1 + b_2\right)\left(\theta\delta\beta_1 + \varepsilon\delta\beta_2\right) + 1$$

Routh Hurwitz stability criterion accomplished so if $\Re_0 < 1$ all the eigenvalues from the linear equilibrium system are negative which generate the free equilibrium point of disease $E_0\left(S, E, I_1, I_2\right)$ locally stability asymptotic. As a result of this prove to be the case that point of any of the free equilibrium disease $E_0\left(S, E, I_1, I_2\right)$ asymptotic stability it means that the disease not spread to other individuals.

3.4 *Stability of Endemic Equilibrium Point*

Theorem 2. The endemic equilibrium point $E_1 = \left(S^*, E^*, I_1^*, I_2^*\right)$ will locally stability asymptotic if

$\Re_0 > 1$ and

$\mu\varepsilon\left(b_2\omega_2 + b_0 b_2 + b_0\omega_2\right) + \theta\left(\mu b_2^2 + \varepsilon\delta\beta_2\right) > \mu\omega_1\left(\varepsilon\omega_2 + \theta b_2\right) + \varepsilon^2\delta\beta_2$

Proof:

We use *the manifold center* theorem to purposed stability from endemic equilibrium if $\Re_0 > 1$. The solution of point equilibrium on the model assumed that occur around a point bifurcation, when $\Re_0 = 1$. β_1 as bifurcation parameter of the equation of $\Re_0$ with $\beta_1 = \beta_1^*$, obtained.

$$\beta_1 = \frac{-\beta_2\delta\theta\omega_1 - \beta_2\delta b_1\varepsilon - \mu b_0\omega_1\omega_2 + \mu b_0 b_1 b_2}{\delta\left(b_2\theta + \omega_2\varepsilon\right)}$$

Obtained a Jacobian matrix as follows $\beta_1 = \beta_1^*$ is

$$J\left(E_0, \beta_1^*\right) = \begin{bmatrix} -\mu & 0 & -\dfrac{-\beta_2\delta\theta\omega_1 - \beta_2\delta b_1\varepsilon - \mu b_0\omega_1\omega_2 + \mu b_0 b_1 b_2}{\mu\left(b_2\theta + \omega_2\varepsilon\right)} & -\dfrac{\beta_2\delta}{\mu} \\ 0 & -b_0 & \dfrac{-\beta_2\delta\theta\omega_1 - \beta_2\delta b_1\varepsilon - \mu b_0\omega_1\omega_2 + \mu b_0 b_1 b_2}{\mu\left(b_2\theta + \omega_2\varepsilon\right)} & \dfrac{\beta_2\delta}{\mu} \\ 0 & \theta & -b_1 & \omega_2 \\ 0 & \varepsilon & \omega_1 & -b_2 \end{bmatrix}$$

Eigenvalues can be obtained from $\left|\lambda I - J\left(E_0, \beta_1^*\right)\right| = 0$. Obtained $\lambda_1 = -\mu, \lambda_2 = 0$ and the roots of $\lambda^2 + x_1\lambda + x_2 = 0$

A right eigenvector that corresponded to the value of eigen $\lambda_2 = 0$ denotes with $\boldsymbol{w} = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \end{bmatrix}^T$. Right eigenvector $\boldsymbol{w}$ fulfill

$$\begin{bmatrix} -\mu w_1 - \dfrac{\left(-\beta_2\delta\theta\omega_1 - \beta_2\delta b_1\varepsilon - \mu b_0\omega_1\omega_2 + \mu b_0 b_1 b_2\right) w_3}{\mu\left(b_2\theta + \omega_2\varepsilon\right)} - \dfrac{\beta_2\delta\omega_4}{\mu} \\ -b_0 w_2 + \dfrac{\left(-\beta_2\delta\theta\omega_1 - \beta_2\delta b_1\varepsilon - \mu b_0\omega_1\omega_2 + \mu b_0 b_1 b_2\right)\omega_3}{\mu\left(b_2\theta + \omega_2\varepsilon\right)} + \dfrac{\beta_2\delta\omega_4}{\mu} \\ \theta w_2 - b_1\omega_3 + \omega_2 w_4 \\ \varepsilon w_2 + \omega_1\omega_3 - b_2 w_4 \end{bmatrix} = 0$$

taken $w_3 = 1$ and $w_4 = 1$ so obtained

$$w_2 = \frac{-\omega_1 + b_2}{\varepsilon} \text{ dan } w_1 = -\frac{-\beta_2\delta\theta\omega_1 - \beta_2\delta\varepsilon b_1 - \mu b_0\omega_1\omega_2 + \mu b_0 b_1 b_2 + \beta_2\delta\varepsilon\omega_2 + \beta_2\delta\theta b_2}{\mu^2\left(\varepsilon\omega_2 + \theta b_2\right)},$$

so the right eigenvector is

$$\boldsymbol{w} = \begin{bmatrix} \dfrac{-\beta_2\delta\theta\omega_1 - \beta_2\delta\varepsilon b_1 - \mu b_0\omega_1\omega_2 + \mu b_0 b_1 b_2 + \beta_2\delta\varepsilon\omega_2 + \beta_2\delta\theta b_2}{\mu^2\left(\varepsilon\omega_2 + \theta b_2\right)} \\ \dfrac{-\omega_1 + b_2}{\varepsilon} \\ 1 \\ 1 \end{bmatrix}^T.$$

Further, a left eigenvector that corresponded to the value of eigen $\lambda_2 = 0$ denotes with $\boldsymbol{v} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$. Left eigenvector $\boldsymbol{v}$ fulfill

$$\mathbf{v}\left(J\left(E_0, \beta_1^*\right)\right) = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \begin{bmatrix} -\mu & 0 & -\dfrac{-\beta_2\delta\theta\omega_1 - \beta_2\delta b_1\varepsilon - \mu b_0\omega_1\omega_2 + \mu b_0 b_1 b_2}{\mu\left(b_2\theta + \omega_2\varepsilon\right)} & -\dfrac{\beta_2\delta}{\mu} \\ 0 & -b_0 & \dfrac{-\beta_2\delta\theta\omega_1 - \beta_2\delta b_1\varepsilon - \mu b_0\omega_1\omega_2 + \mu b_0 b_1 b_2}{\mu\left(b_2\theta + \omega_2\varepsilon\right)} & \dfrac{\beta_2\delta}{\mu} \\ 0 & \theta & -b_1 & \omega_2 \\ 0 & \varepsilon & \omega_1 & -b_2 \end{bmatrix} = 0$$

Thus obtained $v_1 = 0$, $v_2 = \dfrac{v_4\mu\left(\varepsilon\omega_2 + \theta b_2\right)}{\beta_2\delta\theta + \omega_2\mu b_0}$ and $v_3 = \dfrac{v_4\left(-\delta\varepsilon\beta_2 + \mu b_0 b_2\right)}{\beta_2\delta\theta + \omega_2\mu b_0}$.

Then sought $\boldsymbol{v}$ who fulfill $\boldsymbol{v} \cdot \boldsymbol{w} = 1$, so obtained left eigenvalues is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{\varepsilon\mu(\varepsilon\omega_2+\theta b_2)}{\varepsilon\beta_2\delta\theta+\mu\varepsilon\omega_2(b_0+b_2+\varepsilon-\omega_1)+\mu\theta b_2(b_2+\varepsilon-\omega_1)} \\ \dfrac{\varepsilon(\mu b_0 b_2-\varepsilon\delta\beta_2)}{\varepsilon\beta_2\delta(\theta-\varepsilon)+\mu\varepsilon\omega_2(b_0+b_2-\omega_1)+\mu\varepsilon b_0 b_2+\mu\theta b_2(b_2-\omega_1)} \\ \dfrac{\varepsilon(\beta_2\delta\theta+\omega_2\mu b_0)}{\varepsilon\beta_2\delta(\theta-\varepsilon)+\mu\varepsilon\omega_2(b_0+b_2-\omega_1)+\mu\varepsilon b_0 b_2+\mu\theta b_2(b_2-\omega_1)} \end{bmatrix}^T.$$

Suppose that $S = u_1$ $E = u_2$, $I_1 = u_3$, and $I_2 = u_4$. Derivative partial levels of two of the equation at the free equilibrium disease is

$$\frac{\partial^2 f_2}{\partial u_1 \partial u_3} = \frac{\partial^2 f_2}{\partial u_3 \partial u_1} = \beta_1, \quad \frac{\partial^2 f_2}{\partial u_1 \partial u_4} = \frac{\partial^2 f_2}{\partial u_4 \partial u_1} = \beta_2, \quad \frac{\partial^2 f_2}{\partial u_1 \partial \beta_1^*} = u_3 \text{ and } \frac{\partial^2 f_2}{\partial u_3 \partial \beta_1^*} = u_1$$

To determines the kind of a bifurcation parameter at $\Re_0 = 1$, is determined by counting parameter a and b were

$$a = \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial u_i \partial u_j}(u_0, 0) = -\frac{2\varepsilon\left(\delta\beta_2(\varepsilon\omega_2+\theta b_2-\theta\omega_1-\varepsilon b_1)+\mu b_0(b_1 b_2-\omega_1\omega_2)\right)^2}{\mu\delta\left(\mu\varepsilon(b_2\omega_2+b_0 b_2+b_0\omega_2)+\theta(\mu b_2^2+\varepsilon\delta\beta_2)-\mu\omega_1(\varepsilon\omega_2+\theta b_2)-\varepsilon^2\delta\beta_2\right)(\varepsilon\omega_2+\theta\beta_2)}$$

$$b = \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial u_i \partial \beta_2^*} = \frac{\varepsilon\delta(\varepsilon\omega_2+\theta b_2)}{\mu\varepsilon(b_2\omega_2+b_0 b_2+b_0\omega_2)+\theta(\mu b_2^2+\varepsilon\delta\beta_2)-\mu\omega_1(\varepsilon\omega_2+\theta b_2)-\varepsilon^2\delta\beta_2}$$

with the terms $\mu\varepsilon(b_2\omega_2+b_0 b_2+b_0\omega_2)+\theta(\mu b_2^2+\varepsilon\delta\beta_2) > \mu\omega_1(\varepsilon\omega_2+\theta b_2)+\varepsilon^2\delta\beta_2$

According to *Manifold Center* theorem [10], if $a < 0$ and $b > 0$ so locally asymptotically stable. The system of equations will experience bifurcation transcritical at $\Re_0 = 1$, if $a < 0$ and $b > 0$. A system of where it has been shown bifurcation supercritical which is a type of bifurcation transcritical, in bifurcation supercritical there is the exchange of the stability of the in a way that the stability of the equilibrium point of the disease-free and the endemic become stable when $\Re_0 > 1$. So, because $\Re_0 > 1$ of the endemic equilibrium $E_1 = (S^*, E^*, I_1^*, I_2^*)$ locally asymptotically stable.

4. Numerical Results and Discussion

Numerical simulation will be discussed for the spreading of disease tuberculosis. A parameter that used in the simulation is in line with a journal [2,12]. To free disease simulation used parameter value i.e. natural birth rate (δ) is 0.08/365 individual per day, the spread contact from I_1 class (β_1) is 0.11/365 1/individu/day, the spread contact from I_2 class (β_2) is 0.08/365 1/individu/day, natural death rate (μ) is 0.014/365 per day, rate of progression from latent TB class to I_1 class (θ) is 0.09/365 per day, rate of progress from latent TB class to I_2 class (ε) is 0.87/365 per day, rate of the success of treatment active TB in I_1 class (k_1) is 0.09/365 per day, rate of the success of therapy active TB in I_2 class (k_2) is 0.72/365 per day, growth rate of I_1 class to I_2 class (ω_1) is 0.89/365 per day, growth rate of I_2 class to I_1 class (ω_2) is 0.069/365 per day, death rate by disease in I_1 class (d_1) is 0.2/365 per day and death rate by disease in I_2 class (d_2) is 0.05/365 per day.

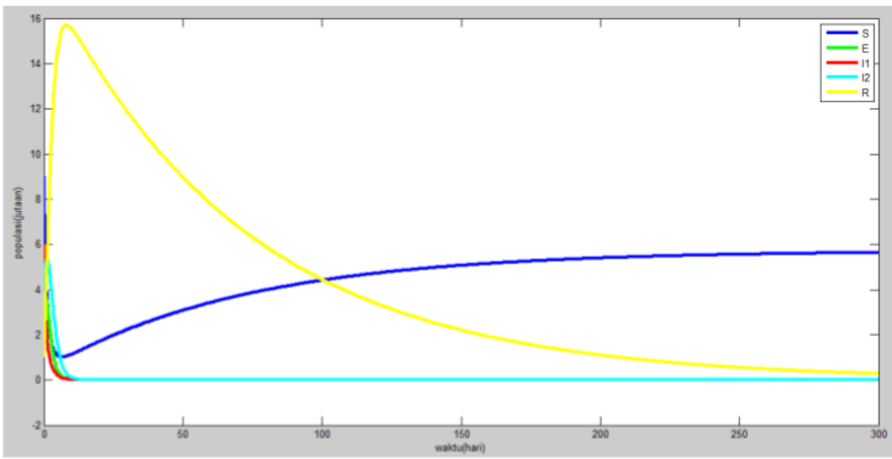

Figure 1 Free disease simulation graphic

The graphic at Figure 1, with initial condition $S(0), E(0), I_1(0), I_2(0), R(0) = 9,2,6,3,1$ indicate the changes in the number of individuals at subpopulation susceptible, latent, infectious treated at home, infectious treated at hospital and recovery from $t = 0$ to $t = \infty$. At the initial condition, individuals at susceptible class are 9 (million unit) then decrease to 1 then increase and stable at $t > 250$. Then the number of latent individuals initially about 2 (million unit) decreased nearly 0 and experiencing constant changes. While many individuals classes are infected treatment at home and hospitals have changed much like on class latent and constantly changed because all three are the same exposed to TB disease the only difference is in the class latent not can transmit or spread TB disease. An individual quantity of recovery class of the lost or increases but its will decreased at *t>25*. Due to a change in the number at subpopulation closely resembled those of the state of balances endemic disease TB $E_1 = (5.714285714, 0, 0, 0)$ then an endemic point equilibrium E_0 of the system is asymptotically stable.

To endemic simulation used parameter value, i.e., natural birth rate (δ) is 0.17/365 individual per day, the spread contact from I_1 class (β_1) is 0.41/365 1/individu/day, the spread contact from I_2 class (β_2) is 0.2077/365 1/individu/day, natural death rate (μ) is 0.014/365 per day, rate of progression from latent TB class to I_1 class (θ) is 0.56/365 per day, rate of progression from latent TB class to I_2 class (ε) is 0.0294/365 per day, rate of the success of treatment active TB in I_1 class (k_1) is 0.2906/365 per day, rate of the success of treatment active TB in I_2 class (k_2) is 0.72/365 per day, growth rate of I_1 class to I_2

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class (ω_1) is 0.89/365 per day, growth rate of I_2 class to I_1 class (ω_2) is 0.069/365 per day, death rate by disease in I_1 class (d_1) is 0.2/365 per day and death rate by disease in I_2 class (d_2) is 0.05/365 per day.

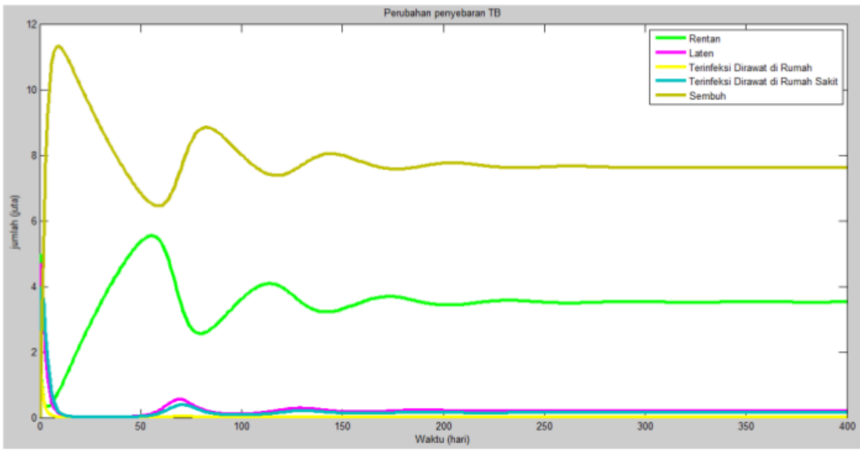

Figure 2 Endemic simulation graphic

The graphic at Figure 3.2 with initial condition $S(0), E(0), I_1(0), I_2(0), R(0) = 5,4,3,2,0$ indicate the changes in the number of individuals at subpopulation susceptible, latent, infectious treated at home, infectious treated at hospital and recovery from $t = 0$ to $t = \infty$. At the initial condition, individuals at susceptible class are 5 (million unit) then decrease to 0 and fluctuation keep moving to show that susceptible individuals begin infected by TB disease, at $t > 250$ experiencing constant changes with the number approaching 3.5135 (million unit). Then the number of latent individuals initially about 4 (million unit). The number of latent individuals decreased. But after reached its peak, the number of latent individuals increased at $t > 50$, and fluctuation keeps moving till $t > 200$ started changes constant of nearly 0.1999 (million unit). While a number of individuals classes are infected are treated at home and at hospitals have changes much like on class latent and changed constant of 0.0113 (million unit) and 0.1430 (million unit) because all three the same exposed to TB disease the only difference is in the class latent not can transmit or spread TB disease. An individual quantity of a class of recovery of the lost or increases due at the same time individuals are infected are treated in the house of and of individuals are infected are treated in the hospital has experienced a fall in.But after the was at its peak, a characteristic quantity of recovery of the lost or is experiencing a few its dips and having the rate at which constant with the total number of closely resembled those of the 7.6071 (million unit). Due to a change in the number at subpopulation closely resembled those of the state of balances endemic disease TB $E_1 = (3.513204517, 0.2002239588, 0.01130836150, 0.1432471966, 7.601727956)$ then a point equilibrium endemic E_1 of the system is asymptotically stable.

5. Conclusion

Based on the results of the analysis, we obtained two equilibrium points, i.e., the disease-free and endemic. Disease-free equilibrium is locally asymptotically stable if $\Re_0 < 1$ an additional requirement that the coefficients of the characteristics equation can be fulfilled. While the point of equilibrium endemic is locally asymptotically stable if $\Re_0 > 1$ and additional parameters requirement that bifurcation met, the results show numerical simulations in the case of tuberculosis spread, that was when a population of class vulnerable, latent, infected is treated at home and infected was hospitalized dwindling. While in class recovered a population of increase from the initial condition and at a certain time a population of on each class changed constantly.

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