CHAPTER I
INTRODUCTION

1.1. Background

The interactions between predators and their preys have long fascinated ecologists due to their prevalence throughout nature. Considered among the most extensively researched subjects within the field of ecology, the interaction between predators and prey provides a classic illustration of how populations of living organisms fluctuate over time. A seminal early mathematical concept proposed to describe these fluctuations was the Lotka-Volterra model of predation. Named after the two scientists who developed this initial framework, the model aimed to account for the oscillatory patterns observed in fish stocks inhabiting the Adriatic Sea during the First World War. Despite its simplicity, the Lotka-Volterra equations captured how interdependencies form between opposing groups and how shifts in the numbers of one can create responsive changes in the other. Their formulation established one of the founding template explanations for this ubiquity feedback process so fundamental to the stability of ecological communities (Murray, 2002). The classical work of (Lotka, 1956; Volterra, 1926) has inspired scholars to formulate prey-predator models with more complex relations in their system.

Prey refuge, harvesting, sickness and the Allee effect are very important factors which affect the dynamics of the prey-predator model and can cause very interesting results (González-Olivares et al., 2019; Kumar & Mandal, 2022; Majeed et al., 2019; Min & Wang, 2019; Mukherjee, 2016; M. Sen et al., 2018; Teixeira Alves & Hilker, 2017; Verma & Misra, 2018; Ye et al., 2019).

The study of (D. Sen et al., 2019) have shown that Introducing a Prey Refugee has shown significant potential for fostering stability within ecological communities by limiting encounters between predators and their preys, thereby reducing their interaction frequency. Two common models of refuge have emerged. In one, the amount of refuge increases proportionally with prey numbers, leaving the surplus exposed to predation. Alternatively, a set area serves as a permanent refuge, its fixed capacity unaffected by population fluctuations. Interestingly, while offering additional protection dynamically improves stability, a stationary refuge produces negligible effects on an already balanced

Lotka-Volterra system. By moderating the interdependence between differrent species, even simple refuge mechanics can help stabilize the delicate relationships that easily tip into unstability.

Typically, the impact of prey refuge on the interacting population dynamics is investigated through mathematical models formulated as predator-prey systems. These models incorporate some form of prey refuge that offers protection from predation (Min & Wang, 2019; Pandey et al., 2024; Saha et al., 2018; Sarwardi et al., 2012; Wang et al., 2016) incorporating a constant proportion of prey refuge with a Holling type II response function (Xiao et al., 2018). On the other hand, when refuge is driven proportionally by predator-prey interactions in the Lotka-Volterra model, population outcomes lead to self-limitation, compromising stability (Molla et al., 2022).

Additionally, Allee effects may arise from various biological factors, decreasing anti-predation awareness, genetic drift, reproductive difficulties, or scarce sustenance at low densities (Dittmer & Allee, 1931; Odum & Allee, 1954). A number of studies have examined the effects of incorporating additional Allee factors into population modeling structures that blend Leslie-Gower designs with Holling type II behavioral responses. Specifically, works by (S. Pal et al., 2018, 2019) incorporated amplifying Allee impacts additively into frameworks combining Leslie-Gower approaches and dynamics adhering to Holling type II patterns. Through such integrated theoretical analyses, the authors identified a heightened possibility of localized species extinction. Integrating reinforcing Allee mechanisms synergistically with Lotka-Volterra style competition exhibiting diminishing returns, these inquiries revealed increased vulnerability to localized demise for sparsely spread, small collectives encountering outsized difficulties originating from their low numbers alone before other limiting influences. Accordingly, we propose a model featuring: Holling type II functional response, and additive prey growth depressions representing Allee impacts. Together, these refinements could advance understanding of destabilizing mechanisms threatening community resiliency. The inclusion of these two components will allow us to gain insight into how the Allee effect interacts with prey refuge to shape population dynamics. The proposed model seeks to provide a more comprehensive and realistic representation of real-world predator-prey relationships. Additionally, as the foraging history of prey by predators influences contemporary predator fertility rates, delays naturally occur. Investigations have explored models joining delays in both predator and prey populations amid Allee impacts. These temporary lags were found to significantly sway the stability and mechanics of the system, regularly leading to unpredictable behaviours. Accounting for such intervals provides a more complete characterization of genuine ecological engagements that undeniably involve ancestral foraging consequences. Integrating pauses into our proposed predator-prey structure with Allee impacts and Prey Refugee could yield extra insights into how lags interact with and magnify the impacts of other intricate qualities in the system. This appears a promising path for potential future scholarly work to cultivate an extremely nuanced and representative theoretical framework.

(Anacleto & Vidal, 2020) offering some understandings into the existence of switching stabilities theoretically and inspecting the direction of bifurcation concerning time delay. (Liu, 2014), implies that expanding intervals always causes Hopf-bifurcation in a modified Gower prey-predator system incorporating the Beddington-DeAngelis practical response. The nonlinear incidence rate was proposed by (Gumel & Moghadas, 2003) and used by many other authors (Gaber et al., 2024), it is considered to be more accurate than the simple law of mass action when it comes to disease spreading horizontally in species as it takes into account the crowding effect of the infected individuals.

(Molla et al., 2022) proposed a two-species prey-predator design with prey refuge corresponding to the interplay between predator and prey. An Allee impact was also deemed utilizing a straightforward manifestation of the Allee impact, whilst the predation followed the Holling type II functional response.

The work (Saha and Samanta 2019) analysed an eco-epidemiological model with herd behaviour while the infection followed the functional response of the law of simple mass action and disease in prey incorporating a prey refuge

In this thesis, we go forward as we formulate a new eco-epidemiological model of a prey-predator with an infection in the prey population we also study the effects of the Allee effect using a different term than the one used in (Molla et al., 2022) and we add a prey refuge which is only proportional to the prey as the refugee s considered to be natural hideouts such as branches of trees, caves, camouflages and others, assuming that the number of predators does not affect the number of such hideouts.

The proposed model follows the Holling type II functional response in the predator-susceptible prey interaction while following the law of simple mass action in the predator-infected prey interaction to make the model more realistic as the infection here is assumed to make the infected prey more exposed to the predator and unable to function in the manners that will make hide or escape the predator.

We include the crowding effect of the infected individuals by assuming that the disease is transmitted from infected prey to susceptible prey by contact according to the nonlinear incidence rate that was proposed by (Gumel & Moghadas, 2003).

The significance of this research is that it develops a new mathematical model addressing all the previous factors affecting an eco-epidemiological system, which to our knowledge have never been studied together before. Our research also explores the dynamics of such a system, the resulting dynamics are very interesting in terms of the resulting equilibria and their stability, as well as the extinction scenarios and type of bifurcations that we will get.

The complexity in the interior and planar equilibrium points resulting from the Added Allee and prey refugee effect is also another interesting result of our new model, increasing the number of possible interior equilibrium points.

1.2. Problem Statement

Through this research, we seek to respond to the following:

1. How can we advance a predator-prey model integrating an infection utilizing nonlinear incidences with an additional Allee effect and prey refuge?
2. How can we acquire the equilibrium points for the aforementioned model and conduct the stability analysis?
3. What is the impact of the added Allee effect and prey refuge in stabilizing our eco-epidemiological model?
4. What types of bifurcation may occur in our model and where?

1.3. Problem Restrictions

1. The study is limited to a predator-prey ecological system with a disease spreading in the prey species, leaving the prey species weak and unable to reproduce.
2. The prey compartment is limited by the prey capacity, and affected by the Allee effect, prey refugee, infection, and predation.
3. The Allee effect studied in this research is strong.
4. The prey refugee used in this research is constant.

1.4. Research Objectives

1. Obtain a model that can be more realistic and applicable to different study cases in predator-prey ecological systems, where a disease spreads through the prey species following the non-linear incidence rate with an added Allee effect and prey refugee in susceptible prey.
2. Obtaining the equilibrium points for the model mentioned above and conducting local and global stability analysis for the equilibrium points.
3. Detect the dynamical changes of the added strong Allee and prey refugee on the stability of the system and its complexity.
4. Explore the different types of bifurcation resulting from changing the values of certain bifurcation parameters in our model.

1.5. Benefits of the Research

The benefits of our research can be summarised as the following:

1. The earlier research studying the dynamics of adding an Allee and prey refugee on a prey-predator model doesn't go as far as to study the dynamics of these two factors combined with an infection spreading in prey species, as this case might occur in many ecological systems. The study of such a model becomes significant for developing a mathematical model that can interact with similar ecological systems suffering from a disease.

2. Most ecological models tend to ignore the crowding effect of the infected individuals on the disease spreading. Epidemiological studies stress the impact of the crowding caused by the increasing number of infected individuals, minimizing the speed at which a disease spread. In this research, we apply the nonlinear incidence rate to our model and examine its effects on such a system.
3. The predator-susceptible prey interaction in our model follows the Holloing type II functional response. Such a response causes the system dynamics to become more complicated, diverging many studies from using it even though such a response gives the model more reliability.

1.6. Systematics of Writing

The systematics of writing this thesis consists of five chapters: Introduction, Literature Review, Research Methods, Discussion, and Closing.

Chapter I "Introduction", consists of background, problem statement, problem restrictions, research objectives, research benefits and writing systematics.

Chapter II "Literature Review", contains supporting theories to write a thesis discussion. Among them are previous studies that significantly inspired our research, definitions of concepts such as mathematical modelling, population dynamics, Malthusian growth model, logistic equations, inter-species competition model, Lotka-Volterra predator-prey model, environmental carrying capacity, Allee effect, prey refugee, nonlinear incidence rate, dynamical system, differential equations, the system of differential equations, equilibrium point, stability of the equilibrium point, bifurcation analysis.

Chapter III "Research Methods", explains the steps we used to write this thesis. Among them are research methods, research procedures, and evaluation.

Chapter IV "Results and Discussion", explains the dynamics of the model of the interaction of the predator with the prey alongside the disease spreading, model formulation, existence and uniqueness of the solution, positivity analysis, solution boundness analysis, equilibrium point analysis, equilibrium point stability analysis, numerical simulation, bifurcation analysis, and comparison between the dynamics of the proposed model and other versions of the model without the Allee effect and prey refugee.

Chapter V "Closing", contains conclusions and suggestions for readers and future researchers.