# DEFAULT PROBABILITY OF MULTIPERIODS COUPON BOND BASED ON CLASSICAL APPROACH

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***Abstract*—Credit risk theory for valuation corporate bond is usually expressed as zero coupon bond. In real bond trading, the most common form of debt is coupon bond. This paper developed model of multiperiods coupon bond with classical approach as default time rule. It means that default occurs when the firm cannot fulfill its payment obligation at coupon date and/or at the maturity date. Some assumptions is used, these are the asset value is log-normally distributed and the universe is risk neutral. The aim of this paper is deriving a fix solution for the value of a multiperiods coupon bond within the framework of the classical model. We used straight forward integration technique and conditional probability theory to derive the equation of default probability. As a result, default probability of the bond at each coupon date is formed in bivariate normal distribution term.**

***Keywords*—conditional probability; bivariate normal distribution; straight forward integration technique**

## I. INTRODUCTION

Credit risk management is one of the most important recent developments in finance industry. It has been an interesting topic of research in the banking and financial community, and has recently attracted the attention of statisticians. Credit risk is the risk resulting from credit events such as changes in credit ratings, restucturing, bankruptcy, etc. Formal denition of credit risk is the distribution of financial losses due to unexpected changes in the credit quality of a counterparty in a financial agreement [1]. Point of view in credit risk is the default event, which happens when the firm can not fulfill its legal obligations in accordance with bond contract.

Merton [2] was the seminal paper which builds a model based on the capital structure of the firm, which becomes the first of the structural model. He assumes that firm is financed by equity and a zero coupon bond with face value $K$ and maturity date $T$. In this approach, the company defaults at the bond maturity date $T$ if its assets has smaller value than the face value of the bond at time $T$.

Up to this time, most corporation tend to issue risky coupon bond rather than a zero coupon bond. At every coupon date until the final payment, the firms have to pay the coupon. At the maturity date, the bondholder receives the face value of the bond plus the last coupon. The bankruptcy of the firm occurs when the firm fails to pay the coupon at the coupon payment and/or the face value of the bond at the maturity date. Geske [3] has derived formulas for valuing coupon bonds. In earlier paper, Geske [4] suggested that when company has coupon bond outstanding, the common stock and coupon bond can be viewed as a compound option. KMV Corporation uses Black & Scholes and Merton methodologies by first converting the debt structure into an equivalent zero coupon bond with maturity one year, for a total promised repayment [5]. Although KMV claims that its methodology can accommodate different classes of debt, they did not explain on how this euquivalent amount is distilled from more complex capital structures [6].

In this paper we valuate multiperiods coupon bond to produce probability of default formula for risky coupon bond with classical default time approach. We construct the formula by straight forward integration.

## II. THEORITICAL FRAMEWORK

### A. *Multiperiods Coupon Bond*

In real bond trading, the most common form of debt is a coupon bond. Suppose the firm has only common stock and coupon bond outstanding. The coupon bond has $n$ interest payments of $c$ dollars each. The firm is assumed to default at coupon date, if the total assets value of the firm is not sufficient to pay the coupon to bondholder. And the firm is assumed to default at the maturity date if the total assets is below the face value of the bond. For this case, if the firm defaults on a coupon date, then all subsequent after that are also default on.

At every coupon date until the final payment, the firm have the option of paying the coupon or turning over the firm to bondholder. The final firm option is to obey the bond contract by paying off the principal at maturity. The financing arrangements for making or missing the interest payments are specified in the indenture conditions of the bond. In Fig 1 we illustrate the cash flow of common corporate coupon bond, that is multiperiods coupon bond [7].

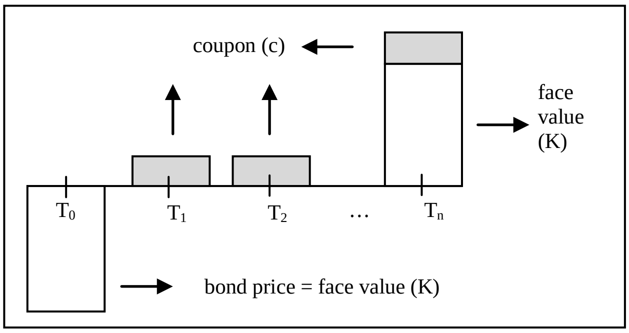


Fig 1. Cash Flow of Multiperiods Coupon Bond

*B. Classical Default Time Approach*

Merton model considers a firm financed through a single debt and single equity issue. The debt form is a zero coupon bond with face value *K* maturing at time *T*. There are no payments until time *T*, and equity holders will accept their rights until *T*, before they decide whether default or not. If the asset value decrease below the face value before *T*, the company still has the possibility to increase the value of assets.

To build this model, we have to make some assumptions for simplifying the analytic solution. Asset value is considering as a Geometric Brownian motion with volatility $\sigma$ and no dividends payments. The equity value of the firm can be determined by Black–Scholes call option formula. Therefore, the model use assumptions of the Black-Scholes option pricing formula which are constant return and volatility, no transaction costs, no dividends, no riskless arbitrage, security trading is continuous, risk free rate is constant for all maturities, and short selling proceeds is permitted [8].

In Merton's model, the default time $\tau$ is a discrete random variable given by:

$$\tau = \begin{cases} T & \text{if} \quad V_T < K \\ \infty & \text{if} \quad V_T \geq K \end{cases} \tag{1}$$

## III. Probability of Default for Multiperiods Coupon Bond

In this model, we assume that the firm can neither repurchase shares nor issue new senior debt. The firm only issued single coupon bond that gives *n* times coupon until the maturity date. At the maturity date, firm has to pay coupon *c* and the face value *K* [9].

Suppose we look at first coupon date $T_1$. If the estimated asset value exceeds or equal the coupon payment of the bond, the bondholder will receive their coupon payment, *c*. The firm is not default at time $T_1$, and the bankruptcy probability for the firm at time $T_1$ follows Theorem 1. Consequently, the firm will default if the asset value is not sufficient to pay the coupon payment.

***Theorem 1*** *Probability of default for multiperiods coupon bond at first coupon date, $T_1$, based on default at maturity approach is*

$$p(\tau = T_1) = N(-G_1) \tag{2}$$

with

$$G_1 = \frac{\ln \dfrac{V_0}{V_1^*} + \left(r - \dfrac{\sigma^2}{2}\right) T_1}{\sigma\sqrt{T_1}}$$

$V_0$ is the value of asset at time 0

$N(x)$ *is univariate standard normal cumulative distribution function.*

**Proof**

Firm equity at coupon date is not equal to zero if $V_{T_1} > V^*$, with $V^*$ is a value of $V_{T_1}$ which is $\left(EQ(V_{T_1}, K_1, T_2 - T_1) - c\right) = 0$. Then based on classical default approach, the default time at ) first coupon date, $\tau$, is a discrete random variable

$$\tau = \begin{cases} \infty & \text{if} \quad V_{T_1} \geq V^* \\ T_1 & \text{if} \quad V_{T_1} < V^* \end{cases} \tag{3}$$

with

$$EQ(V_{T_1}, K_1, T_2 - T_1) = V_{T_1} \; N(d_1)$$
$$- K_1 \exp(-r(T_2 - T_1)) \; N(d_2)$$

Because of $V_t = V_0 \exp\left(\left(\mu - \dfrac{\sigma^2}{2}\right) t + \sigma \, W_t\right)$ with $W_t \sim N(0,t)$, with fixed rate *r*. By risk-neutral valuation then $V_t = V_0 \exp\left(\left(r - \dfrac{\sigma^2}{2}\right) t + \sigma\sqrt{T_1} \; Y\right)$ with $Y \sim N(0,1)$ thus

$$V_{T_1} = V_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) T_1 + \sigma\sqrt{T_1} \; Y\right)$$

Then the probability of default at first coupon date is

$$p(\tau = T_1) = p(V_{T_1} < V^*)$$
$$= p(\ln \; V_{T_1} < \ln \; V^*)$$
$$= p\left(\ln\left(V_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) T_1 + \sigma\sqrt{T_1} \; Y\right)\right) < \ln \; V^*\right)$$

$$= p\left( \ln V_0 + \left( r - \frac{\sigma^2}{2} \right) T_1 + \sigma\sqrt{T_1}\ Y < \ln\ V^* \right)$$

$$= p\left( \sigma\sqrt{T_1}\ Y < \ln\ V^* - \ln V_0 - \left( r - \frac{\sigma^2}{2} \right) T_1 \right)$$

$$= p\left( Y < \frac{\ln \frac{V^*}{V_0} - \left( r - \frac{\sigma^2}{2} \right) T_1}{\sigma\sqrt{T_1}} \right)$$

Because of $Y$ is normally distributed with mean 0 and variance *1*, or $Y \sim N(0,1)$ then

$$p(\tau = T_1) = p\left( Y < \frac{\ln \frac{V^*}{V_0} - \left( r - \frac{\sigma^2}{2} \right) T_1}{\sigma\sqrt{T_1}} \right)$$

$$= N\left( \frac{\ln \frac{V^*}{V_0} - \left( r - \frac{\sigma^2}{2} \right) T_1}{\sigma\sqrt{T_1}} \right)$$

$$= N\left( - \frac{\ln \frac{V_0}{V^*} + \left( r - \frac{\sigma^2}{2} \right) T_1}{\sigma\sqrt{T_1}} \right)$$

$$= N(-G_1)$$

with

$$G_1 = \frac{\ln \frac{V_0}{V^*} + \left( r - \frac{\sigma^2}{2} \right) T_1}{\sigma\sqrt{T_1}}$$

and $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left( -\frac{s^2}{2} \right) ds$ is univariate standard normal cumulative distribution function ∎

Then we look at the second coupon date *$T_2$*. The decision of defaulted firm is as the same for the first coupon date. If the estimated asset value exceeds or equal the coupon payment of the bond, the bondholder will receive their coupon payment, *c*. The firm is not default at time *$T_2$*, and the bankruptcy probability for the firm at time *$T_2$* follows Theorem 2. Consequently, the firm will default if the asset value is not sufficient to pay the coupon payment.

**Theorem 2** *Probability of default for multiperiods coupon bond at the second coupon date, $T_2$, based on default at maturity approach is*

$$p(\tau = T_2) = 1 - \frac{N_2(G_1, G_2, \rho_{1,2})}{N(G_1)} \tag{4}$$

*with*

$$G_1 = \frac{\ln \frac{V_0}{V_1^*} + \left( r - \frac{\sigma^2}{2} \right) T_1}{\sigma\sqrt{T_1}}$$

$$G_2 = \frac{\ln \frac{V_0}{V_2^*} + \left( r - \frac{\sigma^2}{2} \right) T_2}{\sigma\sqrt{T_2}}$$

$$\rho = \frac{\sqrt{T_1}}{\sqrt{T_2}}$$

$N(x)$ *is univariate standard normal cumulative distribution function and* $N_2(x, y\ ; \rho)$ *is bivariate standard normal cumulative distribution function with correlation coefficient* $\rho$ *.*

**Proof**

Probability of default for multiperiods coupon bond at the second coupon date is conditional probability for firm not default at $T_2$ if the firm is not default at time $T_1$.

Probabability for not default at $T_1$ is $p\left(V_{T_1} \geq V_1^*\right)$. Then probability of default at $T_2$ if the firm is not default at time $T_1$ is

$$p\left(V_{T_2} < K_1 \mid V_{T_1} \geq V_1^*\right) \tag{5}$$

The solution for eq (5) is

$$p\left(V_{T_2} < K_1 \mid V_{T_1} \geq V_1^*\right) = 1 - p\left(V_{T_2} \geq K_1 \mid V_{T_1} \geq V_1^*\right)$$

$$= 1 - \frac{N_2(G_1, G_2, \rho_{1,2})}{N(G_2)} \tag{6}$$

with

$$G_1 = \frac{\ln \frac{V_0}{V_1^*} + \left( r - \frac{\sigma^2}{2} \right) T_1}{\sigma\sqrt{T_1}}$$

$$G_2 = \frac{\ln \frac{V_0}{V_2^*} + \left( r - \frac{\sigma^2}{2} \right) T_2}{\sigma \sqrt{T_2}}$$

$$\rho = \frac{\sqrt{T_1}}{\sqrt{T_2}}$$

$N(x)$ is univariate standard normal cumulative distribution function and $N_2(x, y ; \rho)$ is bivariate standard normal cumulative distribution function with correlation coefficient $\rho$. ■

**Theorem 3** *Probability of default for multiperiods coupon bond at the third coupon date, $T_3$, based on default at maturity approach is*

$$\frac{p\left(V_{T_3} \geq V_3^*\ ,\ V_{T_2} \geq V_2^*\right)}{p\left(V_{T_2} \geq V_2^*\right)} = \frac{\frac{N_2(G_2, G_3, \rho_{2,3})}{N(G_2)}}{\frac{N_2(G_1, G_2, \rho_{1,2})}{N(G_1)}} \tag{7}$$

with

$$G_1 = \frac{\ln \frac{V_0}{V_1^*} + \left( r - \frac{\sigma^2}{2} \right) T_1}{\sigma \sqrt{T_1}}$$

$$G_2 = \frac{\ln \frac{V_0}{V_2^*} + \left( r - \frac{\sigma^2}{2} \right) T_2}{\sigma \sqrt{T_2}}$$

$$G_3 = \frac{\ln \frac{V_0}{V_3^*} + \left( r - \frac{\sigma^2}{2} \right) T_3}{\sigma \sqrt{T_3}}$$

*$N(x)$ is univariate standard normal cumulative distribution function and $N_2(x, y ; \rho)$ is bivariate standard normal cumulative distribution function with correlation coefficient $\rho$.*

**Proof**

Probability of default for multiperiods coupon bond at the second coupon date is conditional probability for firm not default at $T_3$ if the firm is not default at time $T_2$.

Probabability for not default at $T_2$ is $p\left(V_{T_2} \geq V_2^*\right)$. Then probability of default at $T_3$ if the firm is not default at time $T_2$ is

$$p\left(V_{T_2} \geq V_2^* \;\middle|\; V_{T_1} \geq V_1^*\right) = \frac{p\left(V_{T_2} \geq V_2^*\ ,\ V_{T_1} \geq V_1^*\right)}{p\left(V_{T_1} \geq V_1^*\right)} \tag{8}$$

with $V_1^*$ is value of $V_{T_1}$ which satifies $\left(EQ\left(V_{T_1}, V_2^*, T_2 - T_1\right) - c\right) = 0$ and $V_2^*$ is value of $V_{T_2}$ which satisfies $\left(EQ\left(V_{T_2}, V_3^*, T_3 - T_2\right) - c\right) = 0$

Then solution of eq (8) is

$$p\left(V_{T_2} \geq V_2^*\ ,\ V_{T_1} \geq V_1^*\right)$$
$$= p\left(\ln\ V_{T_2} \geq \ln\ V_2^*\ ,\ \ln\ V_{T_1} \geq \ln\ V_1^*\right)$$

Because of $V_{T_1} = V_0 \exp\left(\left( r - \frac{\sigma^2}{2} \right) T_1 + \sigma \sqrt{T_1}\ Y\right)$ with $Y \sim N(0,1)$ then

$$V_{T_2} = V_0 \exp\left(\left( r - \frac{\sigma^2}{2} \right) T_2 + \sigma \sqrt{T_2}\ Z\right) \text{ with } Z \sim N(0,1) \tag{9}$$

By subtitution eq (8) and eq (9) to the eq (7) we get

$$p\left(V_{T_2} \geq V_2^*\ ,\ V_{T_1} \geq V_1^*\right) = p\left(\ln\ V_{T_1} > \ln\ V_1^*\ ,\ \ln\ V_{T_2} > \ln\ V_2^*\right)$$

$$= p\left(\ln\left(V_0 \exp\left(\left( r - \frac{\sigma^2}{2} \right) T_1 + \sigma \sqrt{T_1}\ Y\right)\right) \geq \ln\ V_1^*\right.$$

$$\left.,\ \ln\left(V_0 \exp\left(\left( r - \frac{\sigma^2}{2} \right) T_2 + \sigma \sqrt{T_2}\ Z\right)\right) \geq \ln\ V_2^*\right)$$

$$= p\left(Y \geq \frac{\ln \frac{V_1^*}{V_0} - \left( r - \frac{\sigma^2}{2} \right) T_1}{\sigma \sqrt{T_1}}\ ,\ Z \geq \frac{\ln \frac{V_2^*}{V_0} - \left( r - \frac{\sigma^2}{2} \right) T_2}{\sigma \sqrt{T_2}}\right)$$

$$= p\left(Y < \frac{\ln \frac{V_0}{V_1^*} + \left( r - \frac{\sigma^2}{2} \right) T_1}{\sigma \sqrt{T_1}}\ ,\ Z < \frac{\ln \frac{V_0}{V_2^*} + \left( r - \frac{\sigma^2}{2} \right) T_2}{\sigma \sqrt{T_2}}\right)$$

$$= p\left(Y < G_1\ ,\ Z < G_2\right) \tag{10}$$

with

$$G_1 = \frac{\ln \frac{V_0}{V_1^*} + \left( r - \frac{\sigma^2}{2} \right) T_1}{\sigma \sqrt{T_1}}$$

$$G_2=\frac{\ln\frac{V_0}{V_2^*}+\left(r-\frac{\sigma^2}{2}\right)\,T_2}{\sigma\sqrt{T_2}}$$

Because of $\ln V_{T_1}$ and $\ln V_{T_2}$ is a normal standard distributed random variable, then random variable $Y$ and $Z$ is bivariat standard normal cumulative distribution function with correlation coefficient $\rho=\frac{\sqrt{T_1}}{\sqrt{T_2}}$. Then solution for eq (10) is

$$p\left(Y<G_1\ ,\ Z<G_2\right)=\int_{-\infty}^{G_1}\int_{-\infty}^{G_2}N_2\left(Y,Z,\rho_{1,2}\right)\,dY\,dZ$$

$$=N_2\left(G_1,G_2,\rho_{1,2}\right) \tag{11}$$

with

$$G_1=\frac{\ln\frac{V_0}{V_1^*}+\left(r-\frac{\sigma^2}{2}\right)\,T_1}{\sigma\sqrt{T_1}}$$

$$G_2=\frac{\ln\frac{V_0}{V_2^*}+\left(r-\frac{\sigma^2}{2}\right)\,T_2}{\sigma\sqrt{T_2}}$$

Then to derive the probability of not default at $T_1$ is

$$p\left(V_{T_1}\geq V_1^*\right)=1-p\left(V_{T_1}<V_1^*\right)$$

$$=1-N\left(-\frac{\ln\frac{V_0}{V_1^*}+\left(r-\frac{\sigma^2}{2}\right)\,T_1}{\sigma\sqrt{T_1}}\right)$$

$$=N\left(\frac{\ln\frac{V_0}{V_1^*}+\left(r-\frac{\sigma^2}{2}\right)\,T_1}{\sigma\sqrt{T_1}}\right)$$

$$=N\left(G_1\right) \tag{12}$$

By subtituting eq (11) and (12) to eq (7), then the solution for eq (7) is

$$p\left(V_{T_2}\geq V_2^*\ \middle|\ V_{T_1}\geq V_1^*\right)=\frac{p\left(V_{T_2}\geq V_2^*\ ,\ V_{T_1}\geq V_1^*\right)}{p\left(V_{T_1}\geq V_1^*\right)}$$

$$=\frac{N_2\left(G_1,G_2,\rho_{1,2}\right)}{N\left(G_1\right)} \tag{13}$$

Then probability of default at $T_3$ if the firm is not default at $T_2$.

$$p\left(V_{T_3}<V_3^*\ \middle|\ V_{T_2}\geq V_2^*\right)=1-p\left(V_{T_3}\geq V_3^*\ \middle|\ V_{T_2}\geq V_2^*\right) \tag{14}$$

$$=1-\frac{p\left(V_{T_3}\geq V_3^*\ ,\ V_{T_2}\geq V_2^*\right)}{p\left(V_{T_2}\geq V_2^*\right)} \tag{15}$$

Before deriving eq (15) we have to solve

$$p\left(V_{T_3}\geq V_3^*\ ,\ V_{T_2}\geq V_2^*\right)$$

With the same technique as Theorem 2, we have

$$p\left(V_{T_3}\geq V_3^*\ ,\ V_{T_2}\geq V_2^*\right)=\frac{N_2\left(G_2,G_3,\rho_{2,3}\right)}{N\left(G_2\right)} \tag{16}$$

with

$$G_2=\frac{\ln\frac{V_0}{V_2^*}+\left(r-\frac{\sigma^2}{2}\right)\,T_2}{\sigma\sqrt{T_2}}$$

$$G_3=\frac{\ln\frac{V_0}{V_3^*}+\left(r-\frac{\sigma^2}{2}\right)\,T_3}{\sigma\sqrt{T_3}}$$

By subtituting eq (13) and (16), then eq (15) will be

$$\frac{p\left(V_{T_3}\geq V_3^*\ ,\ V_{T_2}\geq V_2^*\right)}{p\left(V_{T_2}\geq V_2^*\right)}=\frac{\frac{N_2\left(G_2,G_3,\rho_{2,3}\right)}{N\left(G_2\right)}}{\frac{N_2\left(G_1,G_2,\rho_{1,2}\right)}{N\left(G_1\right)}} \tag{17}$$

with

$$G_1=\frac{\ln\frac{V_0}{V_1^*}+\left(r-\frac{\sigma^2}{2}\right)\,T_1}{\sigma\sqrt{T_1}}$$

$$G_2=\frac{\ln\frac{V_0}{V_2^*}+\left(r-\frac{\sigma^2}{2}\right)\,T_2}{\sigma\sqrt{T_2}}$$

$$G_3=\frac{\ln\frac{V_0}{V_3^*}+\left(r-\frac{\sigma^2}{2}\right)\,T_3}{\sigma\sqrt{T_3}}$$

With Theorem 1, 2, and 3, we have conclusion:

*Probability of default for multiperiods coupon bond at the i-th coupon date, $T_i$, based on default at maturity approach is*

$$\frac{\dfrac{N_2\left(G_{i-1}, G_i, \rho_{i-1,i}\right)}{N\left(G_{i-1}\right)}}{\dfrac{N_2\left(G_{i-2}, G_{i-1}, \rho_{i-2,i-1}\right)}{N\left(G_{i-2}\right)}} \tag{18}$$

## IV. CONCLUSION

This paper is about developing a model of multiperiods coupon bond based on classical approach default time. This paper aim is to find a closed-form solution for the default probability at each coupon date. It conclusion is default probability of the bond at each coupon date is formed in bivariate normal distribution term.


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